The Carathéodory-Fejér Interpolation Problems and the von-Neumann Inequality

A Dissertation
submitted in partial fulfilment of the requirements
for the award of the degree of

Doctor of Philosophy

by

Rajeev Gupta

Department of Mathematics
Indian Institute of Science
Bangalore - 560012

July 2015

To
my parents
and
my elder brother

Declaration

I hereby declare that the work reported in this thesis is entirely original and has been carried out by me under the supervision of Prof. Gadadhar Misra at the Department of Mathematics, Indian Institute of Science, Bangalore. I further declare that this work has not been the basis for the award of any degree, diploma, fellowship, associateship or similar title of any University or Institution.

Rajeev Gupta
S. R. No. 6910-110-101-07875

Indian Institute of Science,
Bangalore,
July, 2015.

Prof. Gadadhar Misra
(Research advisor)

Acknowledgement

I take this opportunity to thank my PhD supervisor, Prof. Gadadhar Misra, for his enthusiasm, motivation, immense patience and continuous support throughout my stay at Indian Institute of Science. He bestowed an ample amount of time for many long and fruitful discussions, despite being the Chairman of the department. His joy and enthusiasm towards research was contagious, thereby encouraging me to explore the subject with aggression and novelty. A special thanks is due for his meticulous scrutiny of this thesis draft and a major reorganization of the same. His support and guidance not only helped me grow as a researcher but also shaped me into a better human being.

I would like to thank Prof. Pisier for some very insightful discussions during IWOTA 2013. I would like to thank him and Prof. Paulsen for some excellent lectures at the Institute for Mathematical Sciences, Chennai.

I would like to thank the faculty members at the Department of Mathematics in IISc, especially for offering such a wide variety of courses. The courses offered in topics in operator theory by Prof. Gadadhar Misra have benefited me immensely. I would also like to thank Prof. Manjunath Krishnapur for allowing me to simultaneously attend and assist in the probability theory course.

The primary reason for my embracing research in mathematics is the summer program MTTS. I would like to mention a special thanks to Prof. Kumaresan for organizing the same.

I am extremely thankful to the administrative staff at the Department of Mathematics for seeing me through the many complicated formalities that have constantly risen during my stint at IISc.

The many discussions in mathematics with Kartick, Nanda, Pranav, Ramiz, Samya, Tulasi and Vikram have been very rewarding. A heartfelt thanks to all of them. Kartick, Pranav and Sayani deserve a special mention for pinpointing the mistakes in latex that emerged while penning this thesis.

My life at IISc has been enjoyable, pleasant and colourful because of my coexistence with folks like Arpan, Atreyee, Avijit, Bidyut, Dr. DD, Kartick, Mizanur, Nirupam, Pranav, Ramiz, Sayan, Sayani, Tulasi and Vikram. It has only been more cheerful because of friends like Amardeep, Bidhan, Manojit, Nanda, Rakesh, Samrat, Somnath, Sameer and Soumitra. It would be unfair not to mention Jiggy and Paddhu for they have made my life here very charming.

I acknowledge the National Board for Higher Mathematics (NBHM), Government of India for providing me the funding and fellowship to pursue research at IISc.

This acknowledgement would be incomplete without thanking my family for the support they have provided me all through. No words can thank my father and my elder brother for being such an immovable shield for me at times of difficulty. Without their love, hard work and encouragement, this thesis would not have been possible.

Abstract

The validity of the von-Neumann inequality for commuting n - tuples of 3×3 matrices remains open for $n \geq 3$. We give a partial answer to this question, which is used to obtain a necessary condition for the Carathéodory-Fejér interpolation problem on the polydisc $\mathbb{D}^n$. In the special case of $n = 2$ (which follows from Ando's theorem as well), this necessary condition is made explicit.

An alternative approach to the Carathéodory-Fejér interpolation problem, in the special case of $n = 2$, adapting a theorem of Korányi and Pukánzsky is given. As a consequence, a class of polynomials are isolated for which a complete solution to the Carathéodory-Fejér interpolation problem is easily obtained. A natural generalization of the Hankel operators on the Hardy space of $H^2(\mathbb{T}^2)$ then becomes apparent. Many of our results remain valid for any $n \in \mathbb{N}$, however, the computations are somewhat cumbersome for $n > 2$ and are omitted.

The inequality $\lim_{n\to\infty} C_2(n) \leq 2K_G^{\mathbb{C}}$, where $K_G^{\mathbb{C}}$ is the complex Grothendieck constant and

$$C_2(n) = \sup\left\{\|p(\boldsymbol{T})\| : \|p\|_{\mathbb{D}^n,\infty} \leq 1, \|\boldsymbol{T}\|_\infty \leq 1\right\}$$

is due to Varopoulos. Here the supremum is taken over all complex polynomials p in n variables of degree at most 2 and commuting n - tuples $\boldsymbol{T} := (T_1, \ldots, T_n)$ of contractions. We show that

$$\lim_{n\to\infty} C_2(n) \leq \frac{3\sqrt{3}}{4} K_G^{\mathbb{C}}$$

obtaining a slight improvement in the inequality of Varopoulos.

We show that the normed linear space $\ell^1(n)$, $n > 1$, has no isometric embedding into $k \times k$ complex matrices for any $k \in \mathbb{N}$ and discuss several infinite dimensional operator space structures on it.

Contents

1 Introduction

The fundamental inequality of von-Neumann saying that $\|T\| \le 1$ if and only if $\|p(T)\| \le \|p\|_{\mathbb{D},\infty}$ for any polynomial p, has lead to several new developments in modern operator theory. This inequality follows from the Sz.-Nagy dilation theorem, indeed, it is equivalent to it. The homomorphisms $\rho : \mathbb{C}[Z] \to \mathscr{B}(\mathbb{H})$, where $\mathbb{C}[Z]$ is the polynomial ring and $\mathscr{B}(\mathbb{H})$ is the algebra of bounded linear operators, on some complex separable Hilbert space $\mathbb{H}$, are clearly in one-one correspondence with operators T in $\mathscr{B}(\mathbb{H})$. Thus given $T \in \mathscr{B}(\mathbb{H})$, one defines the homomorphism $\rho_T(p) = p(T)$ and conversely given ρ, one may set $T := \rho(z)$. An equivalent formulation of the von-Neumann inequality is the statement: A homomorphism ρ is contractive, that is, $\|\rho(p)\| \le \|p\|_{\mathbb{D},\infty}$ for all $p \in \mathbb{C}[Z]$ if and only if $\|T\| := \|\rho(z)\| \le 1$.

The Sz.-Nagy dilation theorem for a homomorphism ρ is the statement:

The homomorphism ρ is contractive if and only if there exists a Hilbert space $\mathbb{K}$ containing $\mathbb{H}$ and a $*$-homomorphism $\tilde{\rho} : C(\mathbb{T}) \to \mathscr{B}(\mathbb{K})$ such that

$$P_{\mathbb{H}} \tilde{\rho}(p)_{|\mathbb{H}} = \rho(p),\ p \in \mathbb{C}[Z].$$

Since $\sigma(\tilde{\rho}(z)) \subset \mathbb{T}$ and $\tilde{\rho}$ is a $*$-homomorphism, it follows that

$$\|\rho(p)\| \le \|P_{\mathbb{H}} \tilde{\rho}(p)_{|\mathbb{H}}\| \le \|\tilde{\rho}(p)\| \le \|p\|_{\mathbb{T},\infty} \le \|p\|_{\mathbb{D},\infty},$$

which is the von-Neumann inequality. The existence of the $*$-homomorphism $\tilde{\rho}$ can be obtained, among several other methods, following the Schaffer construction of the (unitary power) dilation.

Over the past five or six decades, the question of the von-Neumann inequality and the Sz.-Nagy dilation has been studied vigorously. In explicit terms, these two questions are stated below. Let $\mathbb{C}[Z_1, \ldots, Z_n]$ denote the ring of complex valued polynomials in n variables.

(1) If $T_1, \ldots, T_n$ is a tuple of commuting contractions, does it follow that $\|p(T_1, \ldots, T_n)\| \le \|p\|_{\mathbb{D}^n,\infty}$ for any polynomial $p \in \mathbb{C}[Z_1, \ldots, Z_n]$?

(2) If ρ is a contractive homomorphism, that is, $\|\rho(p)\| \le \|p\|_{\mathbb{D}^n,\infty}$, $p \in \mathbb{C}[Z_1,\ldots,Z_n]$, does it follow that $\rho(p) = P_{\mathbb{H}}\tilde{\rho}(p)_{|_{\mathbb{H}}}$ for some $*$-homomorphism $\tilde{\rho} : C(\mathbb{T}^n) \to \mathscr{B}(\mathbb{K})$, where $\mathbb{K}$ is some Hilbert space containing $\mathbb{H}$?

As is well known, via the foundational work of Arveson [Arv69, Arv72], the second question is equivalent to the complete contractivity of the homomorphism ρ:

$$\|\rho(P)\| \le \|P\|^{\mathrm{op}}_{\mathbb{D}^n,\infty}, \text{ where } P = \big(\!\big(p_{ij}\big)\!\big),\ p_{ij} \in \mathbb{C}[Z_1,\ldots Z_n]$$

$$\text{and } \|P\|^{\mathrm{op}}_{\mathbb{D}^n,\infty} = \sup\big\{\big\|\big(\!\big(p_{ij}\big)\!\big)\big\| : z \in \mathbb{D}^n\big\}.$$

If $n = 1$, as we have seen, an affirmative answer to both of these questions are obtained via the von-Neumann inequality and the Sz.-Nagy dilation theorem. Indeed, an affirmative answer to either of these questions gives an affirmative answer to the other. This continues to be the case even if $n = 2$, thanks to the celebrated theorem of Ando. However for $n = 3$, examples due to Varopoulos-Kaijser and Parrott show that neither (1) nor (2) has an affirmative answer.

Varopoulos, in a second paper, showed that

$$K_G^{\mathbb{C}} \le \sup\|p(T_1,\ldots,T_n)\| \le 2K_G^{\mathbb{C}}, \tag{1.1}$$

where $K_G^{\mathbb{C}}$ denote the complex Grothendieck constant and supremum is over all $n \in \mathbb{N}$, tuples of commuting contractions $T = (T_1,\ldots,T_n)$ and polynomials p of degree 2 with $\|p\|_{\mathbb{D}^n,\infty} \le 1$. He lamented if 2 appearing on the right hand side of this inequality, can be removed. The examples due to Varopoulos leaves the following question (cf. [Pis01, Chapter 1, Page 24] open:

Question 1.1. For a fixed $n \in \mathbb{N}, n \ge 3$ and $M > 0$, does there exist a commuting contractive n - tuple of operators $T_1,\ldots,T_n$ such that

$$\sup_{p\in\mathbb{C}[Z_1,\ldots,Z_n]} \frac{\|p(T_1,\ldots,T_n)\|}{\|p\|_{\mathbb{D}^n,\infty}} > M.$$

A class of homomorphism, which include the example of Parrott were studied further in [Mis94, Pau92], where the question of contractivity vs. complete contractivity of these homomorphism was reduced to certain linear maps. The reason for this lies in showing that the contractivity(respectively complete contractivity) of these homomorphisms is determined by their restriction to the linear polynomials. To explain this in some detail and for use throughout this thesis, we introduced the following notations,

Let Ω be a bounded and polynomially convex domain in $\mathbb{C}^n$. Let $\mathscr{A}(\Omega)$ be the completion of $\mathbb{C}[Z_1,\ldots,Z_n]$ with respect to norm $\|\cdot\|_{\Omega,\infty}$, where $\|f\|_{\Omega,\infty}=\sup\{|f(\omega)|:\omega\in\Omega\}$ for every $f\in\mathbb{C}[Z_1,\ldots,Z_n]$. Let $\mathscr{P}_k[Z_1,\ldots,Z_n]$ denote the set of all polynomials in n variables of degree at most k. When number of variables is clear from the context we omit the variables $Z_1,\ldots,Z_n$. Let $H^\infty(\Omega)$ denote the set of all complex valued bounded holomorphic functions on Ω and $\mathbb{D}$ be the unit disc in $\mathbb{C}$. For each $\omega\in\Omega$, set

$$H^\infty(\Omega,\mathbb{D})=\left\{f\in H^\infty(\Omega):\|f\|_{\Omega,\infty}\le 1\right\} \text{ and } H^\infty_\omega(\Omega,\mathbb{D})=\left\{f\in H^\infty(\Omega,\mathbb{D}):f(\omega)=0\right\}.$$

Let $T=(T_1,\ldots,T_n)$ be a tuple of bounded operators on some fixed separable Hilbert space $\mathbb{H}$ and $\omega=(\omega_1,\ldots,\omega_n)$ be a fixed point in Ω. Define the Parrott homomorphism to be the map $\rho_T^{(\omega)}:H^\infty(\Omega)\to\mathscr{B}(\mathbb{H}\oplus\mathbb{H})$ given by the formula

$$\rho_T^{(\omega)}(f)=\begin{pmatrix} f(\omega)I & Df(\omega)\cdot T\\ 0 & f(\omega)I\end{pmatrix},$$

where $Df(\omega)=\left(\frac{\partial}{\partial z_1}f(\omega),\ldots,\frac{\partial}{\partial z_m}f(\omega)\right)$ and $Df(\omega)\cdot T=\frac{\partial}{\partial z_1}f(\omega)T_1+\cdots+\frac{\partial}{\partial z_m}f(\omega)T_m$.

The following lemma, called "the zero lemma", and several of its variants involving functions defined on domains in $\mathbb{C}^n$ and taking values in $k\times k$ matrices have been proved in [Mis84, MNS90, Mis94, Pau92]. The proof below follows closely the one appearing in [Pau92].

Lemma 1.2. *A Parrott homomorphism $\rho_T^{(\omega)}$ is contractive if and only if $\|\rho_T^{(\omega)}(f)\|\le 1$ for all $f\in H^\infty_\omega(\Omega,\mathbb{D})$.*

Proof. Let us assume that $\|\rho_T^{(\omega)}(f)\|\le 1$ for all $f\in H^\infty_\omega(\Omega,\mathbb{D})$. Suppose $g:\Omega\to\mathbb{D}$ is an analytic function and ϕ is the automorphism of $\mathbb{D}$ mapping $g(\omega)$ to 0. Then $\phi\circ g$ is an analytic map from Ω to $\mathbb{D}$ with $(\phi\circ g)(\omega)=0$, therefore $\|\rho_T^{(\omega)}(\phi\circ g)\|\le 1$. Now by von-Neumann's inequality we have $\|\phi^{-1}(\rho_T^{(\omega)}(\phi\circ g))\|\le 1$, which is equivalent to $\|\rho_T^{(\omega)}(g)\|\le 1$. Hence $\rho_T^{(\omega)}$ is a contraction. The converse is trivially true. □

Now, let us assume that Ω is a unit ball in $\mathbb{C}^n$ with respect to some norm and $\omega=0$. Let

$$\mathscr{L}[Z_1,\ldots,Z_n]=\{a_1z_1+\cdots+a_nz_n:a_i\in\mathbb{C}\ \forall i=1\text{ to }n\}$$

be the set of all linear polynomials in m variables. Let ρ_T denote the homomorphism $\rho_T^{(0)}$.

Theorem 1.3. *For the Parrott homomorphism ρ_T, we have*

$$\sup\left\{\|\rho_T(\ell)\|:\ell\in\mathscr{L}[Z_1,\ldots,Z_n],\|\ell\|_{\Omega,\infty}\le 1\right\}=\sup\left\{\|\rho_T(f)\|:f\in H^\infty_0(\Omega,\mathbb{D})\right\}.$$

Proof. If $f \in H_0^\infty(\Omega, \mathbb{D})$ is a holomorphic function, then from the Schwarz lemma [Rud08, Theorem 8.1.2], $\ell := Df(0)$ maps Ω in to the disc of radius $\|f\|_{\Omega,\infty}$ and thus $\|\ell\|_{\Omega,\infty} \leq \|f\|_{\Omega,\infty}$. From the definition of ρ_T, we have $\|\rho_T(\ell)\| = \|\rho_T(f)\|$. Therefore

$$\frac{\|\rho_T(\ell)\|}{\|\ell\|_{\Omega,\infty}} \geq \frac{\|\rho_T(f)\|}{\|f\|_{\Omega,\infty}}$$

and hence

$$\sup\left\{\|\rho_T(\ell)\| : \ell \in \mathscr{L}[Z_1,\ldots,Z_n], \|\ell\|_{\Omega,\infty} \leq 1\right\} \geq \sup\left\{\|\rho_T(f)\| : f \in H_0^\infty(\Omega,\mathbb{D})\right\}.$$

The other inequality is obvious. □

This theorem says that if we wish to establish only the contractivity of Parrott homomorphism ρ_T, it is enough to restrict ρ_T to the linear polynomials.

It also says that if $\Omega = \mathbb{D}^n$, then the Parrott homomorphisms ρ_T are contractive if and only if $T_1,\ldots,T_n$ are contractions. This follows from the Schwarz lemma (cf. [Rud08, Theorem 8.1.2]):

$$\left\{Df(0) \in \mathbb{C}^n : f \in H_0^\infty(\mathbb{D}^n,\mathbb{D})\right\} = \left\{\ell : \ell(\mathbb{D}^n) \subseteq \mathbb{D}\right\}.$$

Consequently, these homomorphisms can not be used to answer the Question 1.1.

We therefore investigate the homomorphism induced by the commuting triple of contractions T_1, T_2, T_3 given by Varopoulos and Kaijser with the property $\|p(T_1,T_2,T_3)\| > \|p\|_{\mathbb{D}^3,\infty}$. This leads to a natural definition of a class of operators which we call Varopoulos operator of type I and II. We investigate the answer to the Question 1.1 assuming the homomorphism ρ_T is induced by T, a tuple of these operators. It is useful to recall that Varopoulos, in a second paper, proved the following.

$$K_G^{\mathbb{C}} \leq \sup \|\rho_{T|\mathscr{P}_2}\| \leq 2K_G^{\mathbb{C}},$$

where $K_G^{\mathbb{C}}$ denote the complex Grothendieck constant and supremum is over all $n \in \mathbb{N}$ and tuples of commuting contractions $T = (T_1,\ldots,T_n)$. Thus it is natural to ask if $\sup_{\|T\|_\infty \leq 1} \|\rho_{T|\mathscr{P}_2}\|$, where $\|T\|_\infty := \max\{\|T_1\|,\ldots,\|T_n\|\}$, is closer to the universal constant $K_G^{\mathbb{C}}$ of Grothendieck than indicated by the inequalities (1.1). We show that inequality on the right can be considerably improved.

Let $\mathbb{H}$ be a separable Hilbert space and $\{e_j\}_{j\in\mathbb{N}}$ be a set of orthonormal basis for $\mathbb{H}$. For any $x \in \mathbb{H}$, define $x^\sharp : \mathbb{H} \to \mathbb{C}$ by $x^\sharp(y) = \sum_j x_j y_j$, where $x = \sum x_j e_j$ and $y = \sum y_j e_j$. For $x, y \in \mathbb{H}$, we set $\left[x^\sharp, y\right] = x^\sharp(y)$.

Definition 1.4 (Varopoulos Operator of Type I (V I))**.** Let $\mathbb{H}$ be a separable Hilbert space. For $x, y \in \mathbb{H}$, define $T_{x,y} : \mathbb{C} \oplus \mathbb{H} \oplus \mathbb{C} \to \mathbb{C} \oplus \mathbb{H} \oplus \mathbb{C}$ by

$$T_{x,y} = \begin{pmatrix} 0 & x^\sharp & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

The operator $T_{x,y}$ will be called Varopoulos operator of type I corresponding to the pair of vectors x, y. If $x = y$ then T_{xy} will simply be denoted by T_x.

Definition 1.5 (Varopoulos Operator of Type II and of order k (V II of order k))**.** Let $\mathbb{H}$ be a separable Hilbert space. For $X \in \mathscr{B}(\mathbb{H})$, let

$$T_X = \begin{pmatrix} 0 & X & 0 & \cdots & 0 \\ 0 & 0 & X & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & X \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

be the operator in $\mathscr{B}(\mathbb{H} \otimes C^{k+1})$. In analogy with the work of Varopoulos [Var76], operators of the form T_X, $X \in \mathscr{B}(\mathbb{H})$, are called Varopoulos operator of type II and of order k.

Let Ω be a bounded domain in $\mathbb{C}^n$ and $\omega \in \Omega$ be a fixed but arbitrary point. As before let $\rho_{xy}^{(\omega)}$ (respectively $\mu_X^{(\omega)}$) denote the induced homomorphism on $H^\infty(\Omega)$, corresponding to a tuple of commuting contractions $\omega_1 I + T_{x_1 y_1}, \ldots, \omega_n I + T_{x_n y_n}$ (respectively $\omega_1 I + T_{X_1}, \ldots, \omega_n I + T_{X_n}$), which is defined as following.

$$\rho_{xy}^{(\omega)}(f) = \begin{pmatrix} f(\omega) & Df(\omega) \cdot x^\sharp & \frac{1}{2} D^2 f(\omega) \cdot A_{x,y} \\ 0 & f(\omega) I & Df(\omega) \cdot y \\ 0 & 0 & f(\omega) \end{pmatrix}$$

for $f \in H^\infty(\Omega)$, where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $x^\sharp = (x_1^\sharp, \ldots, x_n^\sharp)$ and $A_{x,y} = \big(\!\big([x_i^\sharp, y_j]\big)\!\big)$. As the definition of $\rho_{xy}^{(\omega)}(f)$ includes only the terms of order at most 2 from the Taylor series expansion of f, therefore it is quite natural to ask the following question.

Question 1.6. We ask if the contractivity of $\rho_{xy}^{(\omega)}$ on $H^\infty(\Omega)$ is equivalent to contractivity of the restriction to the polynomials of degree at most 2.

Clearly to answer this question, one must first answer a related question generalizing the Carathéodory-Fejér interpolation problem, namely: Given any polynomial p in n variables of degree 2 with $p(0) = 0$, find necessary and sufficient conditions on the coefficients of p to ensure the existence of a holomorphic function h defined on the polydisc $\mathbb{D}^n$ with $h^{(k)}(0) = 0$ for all multi indices k of length at most 2, such that $f := p + h$ maps the polydisc $\mathbb{D}^n$ to the unit disc $\mathbb{D}$.

However the absence of an explicit criterion, in spite of several results which have been obtained recently [FF90, EPP00, Woe02, HWH14], for the solution to this problem for $n > 1$ makes it difficult to answer this question.

We combine a theorem due to Korányi and Pukánszky giving a criterion for determining if the real part of a holomorphic function defined on the polydisc $\mathbb{D}^n$ is positive with a theorem due to Parrott to find a solution to the Carathéodory-Fejér interpolation problem. We state these two theorems below.

Theorem 1.7 (Korányi-Pukánszky Theorem). *If the power series $\sum_{\alpha\in\mathbb{N}_0^n} a_\alpha z^\alpha$ represents a holomorphic function f on the polydisc $\mathbb{D}^n$, then $\Re(f(z)) \geq 0$ for all $z \in \mathbb{D}^n$ if and only if the map $\phi : \mathbb{Z}^n \to \mathbb{C}$ defined by*

$$\phi(\alpha) = \begin{cases} 2\Re a_\alpha & \text{if } \alpha = 0 \\ a_\alpha & \text{if } \alpha > 0 \\ a_{-\alpha} & \text{if } \alpha < 0 \\ 0 & \text{otherwise} \end{cases}$$

is positive, that is, the $k \times k$ matrix $\left(\!\left(\phi(m_i - m_j)\right)\!\right)$ is non-negative definite for every choice of $m_1, \ldots, m_k \in \mathbb{Z}^n$.

Let $f : \mathbb{D}^n \to \mathbb{D}$ be a holomorphic function and χ be the Cayley map of the unit disc to the right half plane. Then in the matricial representation of $\phi_{\chi\circ f}$ with respect to the usual order in $\mathbb{Z}^2$, it is not easy to isolate the coefficients of f. We introduce a new order, to be called, the `D-slice` ordering:

Definition 1.8 (D-slice ordering). Suppose $(x_1, y_1) \in P_l$ and $(x_2, y_2) \in P_m$ are two elements in $\mathbb{Z}^2$. Then

1. If $l = m$, then $(x_1, y_1) < (x_2, y_2)$ is determined by the lexicographic ordering on $P_l \subseteq \mathbb{Z}^2$ and
2. if $l < m$ (resp., if $l > m$), then $(x_1, y_1) < (x_2, y_2)$ (resp., $(x_1, y_1) > (x_2, y_2)$).

The matricial representation of the function $\phi_{\chi\circ f}$ is then in the form of a block Toeplitz matrix with respect to the D-slice ordering.

Theorem 1.9 (Parrott's Theorem)**.** *For $i = 1,2$, let $\mathbb{H}_i, \mathbb{K}_i$ be Hilbert spaces and $\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2$, $\mathbb{K} = \mathbb{K}_1 \oplus \mathbb{K}_2$. If*

$$\begin{pmatrix} A \\ C \end{pmatrix} : \mathbb{H}_1 \to \mathbb{K} \text{ and } \begin{pmatrix} C & D \end{pmatrix} : \mathbb{H} \to \mathbb{K}_2$$

are contractions, then there exists $X \in \mathscr{B}(\mathbb{H}_2, \mathbb{K}_1)$ such that $\left(\begin{smallmatrix} A & X \\ C & D \end{smallmatrix}\right) : \mathbb{H} \to \mathbb{K}$ is a contraction.

In this theorem, all the choices for X are given by the formula:

$$(I - ZZ^*)^{1/2} V (I - Y^* Y)^{1/2} - ZC^* Y,$$

where V is a contraction and Y, Z are determined from the formulae:

$$D = (I - CC^*)^{1/2} Y, \; A = Z(I - C^* C)^{1/2}.$$

Our method gives only a (explicit) necessary condition for the existence of a solution to the Carathéodory-Fejér interpolation problem in general. (Surprisingly, for the case of the bi-disc, this necessary condition is exactly the condition for contractivity of the homomorphisms induced by the Varopoulos operators.)

It also gives an algorithm for constructing a solution whenever such a solution exists. The algorithm involves finding, inductively, polynomials p_n of degree at most n such that a certain block Toeplitz operator, made up of multiplication by these polynomials is contractive. A solution to the Carathéodory-Fejér interpolation problem exists if and only if this process is completed successfully.

If $n = 1$ and the necessary condition we have obtained is met, then the algorithm completes successfully and produces a solution to the Carathéodory-Fejér interpolation problem. Thus in this case, we fully recover the solution to the Carathéodory-Fejér interpolation problem.

We also isolate a class of polynomials for which our necessary condition is also sufficient. This is verified using the deep theorem of Nehari reproduced below (cf. [You88, Theorem 15.14, page 194]).

Let $H^2(\mathbb{T})$ denote the Hardy space, a closed subspace of $L^2(\mathbb{T})$. Let P_- denote the orthogonal projection of $L^2(\mathbb{T})$ onto $L^2(\mathbb{T}) \ominus H^2(\mathbb{T})$.

Definition 1.10 (Multiplication Operator)**.** For $\phi \in L^\infty(\mathbb{T})$, we define multiplication operator $M_\phi : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ by $M_\phi(f) = \phi \cdot f$.

Since $\phi \cdot f \in L^2(\mathbb{T})$ for any $\phi \in L^\infty(\mathbb{T})$ and $f \in L^2(\mathbb{T})$ therefore M_ϕ is well defined for all $\phi \in L^\infty(\mathbb{T})$. Also $\|M_\phi\| = \|\phi\|_\infty$ (cf. Theorem 13.14 in [You88]).

Definition 1.11 (Hankel Operator)**.** Let $\phi \in L^\infty(\mathbb{T})$. Hankel operator corresponding to ϕ is the operator $P_- \circ M_\phi|_{H^2(\mathbb{T})}$. It is denoted by H_ϕ.

Theorem 1.12 (Nehari's Theorem)**.** *If $\phi \in L^\infty(\mathbb{T})$ and H_ϕ is the corresponding Hankel operator, then*

$$\inf\{\|\phi - g\|_{\mathbb{T},\infty} : g \in H^\infty(\mathbb{T})\} = \|H_\phi\|_{op}.$$

All this is done for the bi-disc $\mathbb{D}^2$ with the understanding that these computations will go through for the polydisc $\mathbb{D}^n$. Similarly, while we have discussed the Carathéodory-Fejér interpolation problem for polynomials of degree at most 2, again, our methods remain valid for an arbitrary polynomial.

What follows is a detailed description of the results proved in this thesis.

Following [Var76] and [Pis01, Page 24], in chapter 2, we define the quantities:

$$C_k(n) = \sup\left\{\|p(\boldsymbol{T})\| : \|p\|_{\mathbb{D}^n,\infty} \le 1, p \text{ is of degree at most } k, \|\boldsymbol{T}\|_\infty \le 1\right\}$$

and

$$C(n) = \lim_{k\to\infty} C_k(n), \tag{1.2}$$

where $\|\boldsymbol{T}\|_\infty = \max\{\|T_1\|, \dots, \|T_n\|\}$. In this notation, it follows from the von-Neumann inequality and Ando's theorem that $C(1), C(2) = 1$. Also $C_2(3) > 1$, thanks to the example of Varopoulos and Kaijser [Var74] involving an (explicit) homogeneous polynomial of degree 2. Following this, in the paper [Var76], Varopoulos proves the inequality (1.1). Consequently, the limit of the non-decreasing sequence $C_2(n)$ must be bounded below by $K_G^{\mathbb{C}}$. We show that $C_2(3) \ge 1.2$ by means of explicit examples. We were hoping to improve this inequality obtained earlier by Holbrook [Hol01] since our methods appear to be somewhat more direct. In view of the known lower bound for $\lim_{n\to\infty} C_2(n)$ in (1.1), we hoped that the lower bound for $C_2(3)$ itself will be closer to $K_G^{\mathbb{C}}$. In this chapter, we also show that $\|p(T_1, \dots, T_n)\| \le \|p\|_{\mathbb{D}^n,\infty}$ for any n commuting contractions of the form

$$\left\{\left(\begin{pmatrix} \omega_1 & \alpha_1 & 0 \\ 0 & \omega_1 & \beta_1 \\ 0 & 0 & \omega_1 \end{pmatrix}, \dots, \begin{pmatrix} \omega_n & \alpha_n & 0 \\ 0 & \omega_n & \beta_n \\ 0 & 0 & \omega_n \end{pmatrix}\right) : \alpha_i\beta_j = \alpha_j\beta_i, 1 \le i, j \le n, \omega := (\omega_1, \dots, \omega_n) \in \mathbb{D}^n\right\}, \tag{1.3}$$

after assuming that $|\alpha_i| = |\beta_i|$, $1 \le i \le n$. This is interesting considering that the von-Neumann inequality is valid for any commuting n - tuple of 2×2 contractions [MP93, Agl90] and fails for 4×4 contractions [Hol01]. As a corollary of the von-Neumann inequality for a subclass of the operators defined in (1.3), we get the following necessary condition for the Carathéodory-Fejér interpolation problem for the polydisc $\mathbb{D}^n$.

Theorem 1.13. *Let p be a polynomial in n variables of degree 2 such that $p(0) = 0$. There exists a holomorphic function q, defined on polydisc $\mathbb{D}^n$, with $q^{(k)}(0) = 0$, $|k| \le d$ such that $\|p+q\|_\infty \le 1$ only if*

$$\sup_{\|\alpha\|_\infty \le 1} \left\{ \left| \frac{D^2 p(0) \cdot A_\alpha}{2} \right| + \left| Dp(0) \cdot \alpha \right|^2 \right\} \le 1,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $A_\alpha = \left(\!\left(\alpha_i \alpha_j\right)\!\right)$, $D^2 p(0) \cdot A_\alpha = \sum \frac{\partial^2 p}{\partial z_i \partial z_j}(0) \alpha_i \alpha_j$ and $Dp(0) \cdot \alpha = \sum \frac{\partial p}{\partial z_i}(0) \alpha_i$.

We also prove the following theorem, giving a considerable improvement on the upper bound (1.1) previously obtained in [Var76].

Theorem 1.14. $\lim_{n\to\infty} C_2(n) \le \frac{3\sqrt{3}}{4} K_G^{\mathbb{C}}$.

Finally, in this chapter, we investigate in some detail, the contractivity of the homomorphisms $\rho_{x,y}$ induced by the Varopoulos operators of type I (V I) and we come up with the same inequality as in the Theorem 1.13 but for $n = 2$.

In chapter 3, we study the homomorphisms induced by tuples of commuting Varopoulos operators of type II and order 2 and solve the extremal problem (indeed a more general extremal problem obtained in the study of these homomorphism) occurring in the Theorem 1.13 but for $n = 2$. We define

$$p_1(z) = \frac{\partial}{\partial z_1} f(0) + \frac{\partial}{\partial z_2} f(0) z \text{ and } p_2(z) = \frac{1}{2} \frac{\partial^2}{\partial z_1^2} f(0) + \frac{\partial^2}{\partial z_1 \partial z_2} f(0) z + \frac{1}{2} \frac{\partial^2}{\partial z_2^2} f(0) z^2$$

for a holomorphic function f in two variables and we prove the following.

Theorem 1.15. *For $f \in \mathrm{H}_0^\infty(\mathbb{D}^2, \mathbb{D})$, we have,*

$$\sup_{\|X\|_\infty \le 1} \left\| \mathcal{T}\left(Df(0) \cdot X, \frac{1}{2} D^2 f(0) \cdot A_X \right) \right\| = \|\mathcal{T}(M_{p_1}, M_{p_2})\|,$$

where $X = (X_1, X_2)$ is pair of commuting operators, $\|X\|_\infty = \max\{\|X_1\|, \|X_2\|\}$, $A_X = \left(\!\left(X_i X_j\right)\!\right)$ and $\mathcal{T}(A_1, A_2) = \begin{pmatrix} A_1 & A_2 \\ 0 & A_1 \end{pmatrix}$ for any $A_1, A_2 \in \mathscr{B}(\mathbb{H})$.

In the process of solving the extremal problem occurring in this theorem, we reprove the von-Neumann inequality and Ando's theorem for a commuting pair of Varopoulos operators of type II.

In chapter 4, we give an alternative for solving the Carathéodory-Fejér interpolation problem after adapting a theorem of Korányi and Pukánszky. This approach is important

as it is independent of the commutant lifting theorem, whereas the method in chapter 3 strongly depends on it. For a polynomial p in two variables we define

$$p_1(z) = \frac{\partial}{\partial z_1}p(0) + \frac{\partial}{\partial z_2}p(0)z \text{ and } p_2(z) = \frac{1}{2}\frac{\partial^2}{\partial z_1^2}p(0) + \frac{\partial^2}{\partial z_1 \partial z_2}p(0)z + \frac{1}{2}\frac{\partial^2}{\partial z_2^2}p(0)z^2.$$

In the following theorem, we reformulate the Carathéodory-Fejér interpolation problem for the bi-disc $\mathbb{D}^2$.

Theorem 1.16. *For any polynomial p of the form*

$$p(z) = a_{10}z_1 + a_{01}z_2 + a_{20}z_1^2 + a_{11}z_1z_2 + a_{02}z_2^2,$$

there exists a holomorphic function q, defined on the bi-disc $\mathbb{D}^2$, with $q^{(k)}(0) = 0$ for $|k| \le 2$, such that $\|p+q\|_{\mathbb{D}^2,\infty} \le 1$ if and only if $|p_2| \le 1 - |p_1|^2$ and there exists a holomorphic function $f : \mathbb{D} \to \mathscr{B}(L^2(\mathbb{T}))$ with

$$\|f\|_{\mathbb{D},\infty}^{\mathrm{op}} \le 1 \text{ and } \frac{f^{(k)}(0)}{k!} = M_{p_k} \text{ for all } k \ge 0,$$

where $p_0 = 0$ and for $k \ge 3$, $p_k \in \mathbb{C}[Z]$ is a polynomial of degree less than or equal to k. Here M_{p_k} is the multiplication operator on $L^2(\mathbb{T})$ induced by the polynomial p_k.

In this chapter, we show that $|p_1|^2 + |p_2| \le 1$ is a necessary condition for the solution to exist for the Carathéodory-Fejér interpolation problem. In the following theorem, we also isolate a class of polynomials for which this is a sufficient condition for the existence of a solution.

Theorem 1.17. *Let $p_1(z) = \gamma + \delta z$ and $p_2(z) = (\alpha + \beta z)(\gamma + \delta z)$ for some choice of complex numbers α, β, γ and δ. Assume that $|p_1|^2 + |p_2| \le 1$. If either $\alpha\beta\gamma\delta = 0$ or $\arg(\alpha) - \arg(\beta) = \arg(\gamma) - \arg(\delta)$, then $|p_1|^2 + |p_2| \le 1$ is a sufficient condition also for the existence of a solution for the corresponding Carathéodory-Fejér interpolation problem.*

We illustrate, by means of an example, that this necessary condition is not sufficient in general. In the end of this chapter, we give a proof of the Korányi-Pukánszky theorem for the bi-disc using the spectral theorem. This proof can be made to work for the polydisc as well.

In chapter 5, we give a generalization of Nehari's theorem to two variables. In this chapter we define the Hankel operator H_ϕ corresponding to any function $\phi \in L^\infty(\mathbb{T}^2)$. The following theorem shows that the norm of the Hankel operator H_ϕ is the norm of the symbol ϕ with respect to a quotient norm, described below.

Theorem 1.18 (Nehari's theorem for $L^2(\mathbb{T}^2)$)**.** *If* $\phi \in L^\infty(\mathbb{T}^2)$, *then* $\|H_\phi\| = \mathrm{dist}_\infty(\phi, H_1)$.

In this theorem, $H_1 := \left\{ f := \sum_{m+n\geq 0} a_{m,n} z_1^m z_2^n | f \in L^\infty(\mathbb{T}^2) \right\}$ and $\mathrm{dist}_\infty(\phi, H_1)$ is the distance of ϕ from H_1 in $L^\infty-$ norm.

In chapter 6, we study the operators space structures on $\ell^1(n)$. There is a canonical isometric embedding of $\ell^\infty(n)$ into the set of $n \times n$ matrices M_n. However, we show that $\ell^1(n)$, $n > 1$, has no isometric embedding into M_k for any $k \in \mathbb{N}$.

Theorem 1.19. *There is no isometric embedding of* $\ell^1(n)$, $n > 1$, *in to* M_k *for any* $k \in \mathbb{N}$.

The next theorem provides several isometric embeddings of $\ell^1(n)$ into $\mathscr{B}(\mathbb{H})$ for each $n \in \mathbb{N}$.

Let $\mathbb{H}_1, \ldots, \mathbb{H}_n$ be Hilbert spaces and T_i be a contraction on $\mathbb{H}_i$ for $i = 1, \ldots, n$. Assume that the unit circle $\mathbb{T}$ is contained in $\sigma(T_i)$, the spectrum of T_i, for $i = 1, \ldots, n$. Denote

$$\tilde{T}_1 = T_1 \otimes I^{\otimes(n-1)}, \tilde{T}_2 = I \otimes T_2 \otimes I^{\otimes(n-2)}, \ldots, \tilde{T}_n = I^{\otimes(n-1)} \otimes T_n.$$

Theorem 1.20. *Suppose the operators* $\tilde{T}_1, \ldots, \tilde{T}_n$ *are defined as above. Then, the function*

$$f : \ell^1(n) \to \mathscr{B}(\mathbb{H}_1 \otimes \cdots \otimes \mathbb{H}_n)$$

defined by

$$f(a_1, a_2, \ldots, a_n) := a_1 \tilde{T}_1 + a_2 \tilde{T}_2 + \cdots + a_n \tilde{T}_n.$$

is an isometry.

For $n = 2, 3$, we show that all of these embeddings are completely isometric to the MIN structure. In the end of this chapter, using these embeddings and Parrott's example in [Mis94], we construct an operator space structure on $\ell^1(3)$ which is distinct from the MIN structure.

2 Varopoulos Operators of Type I

Let $\mathbb{C}[Z_1, \dots, Z_n]$ denote the set of all polynomials in n complex variables. For every contraction T on a complex Hilbert space, the von-Neumann inequality [vN51] states that $\|p(T)\| \le \|p\|_{\mathbb{D},\infty}$ for every $p \in \mathbb{C}[Z]$. Ando [And63] established an analogous inequality for any two commuting contractions T_1, T_2, namely, $\|p(T_1, T_2)\| \le \|p\|_{\mathbb{D}^2,\infty}$ for every $p \in \mathbb{C}[Z_1, Z_2]$. Varopoulos [Var74] constructed examples showing that the generalization of this inequality to three variables fails. He along with Kaijser also produced an explicit example of three commuting contractions T_1, T_2, T_3 and a polynomial p with the property $\|p(T_1, T_2, T_3)\| > \|p\|_{\mathbb{D}^3,\infty}$. Let $\|\boldsymbol{T}\|_\infty = \max\{\|T_1\|, \dots, \|T_n\|\}$,

$$C_k(n) = \sup\{\|p(\boldsymbol{T})\| : \|p\|_{\mathbb{D}^n,\infty} \le 1, p \text{ is of degree at most } k, \|\boldsymbol{T}\|_\infty \le 1\}. \tag{2.1}$$

and

$$C(n) = \lim_{k\to\infty} C_k(n). \tag{2.2}$$

In this notation, it follows from the von-Neumann inequality and Ando's theorem that $C(1), C(2) = 1$.

Also $C_2(3) > 1$, thanks to the example of Varopoulos and Kaijser [Var74] involving an (explicit) homogeneous polynomial of degree 2. Following this, in the paper [Var76], Varopoulos shows that the limit of the non-decreasing sequence $C_2(n)$ must be bounded above by $2K_G^{\mathbb{C}}$, where $K_G^{\mathbb{C}}$ is the complex Grothendieck constant, the definition is given below. He also showed that the lower bound for this limit is $K_G^{\mathbb{C}}$. Thus he has proved

$$K_G^{\mathbb{C}} \le \lim_{n\to\infty} C_2(n) \le 2K_G^{\mathbb{C}}. \tag{2.3}$$

We show that $C_2(3) \ge 1.2$ by means of explicit examples. We were hoping to improve this inequality obtained earlier by Holbrook [Hol01] since our methods appear to be somewhat more direct. In view of the known lower bound for $\lim_{n\to\infty} C_2(n)$ in (2.3), we hoped that the lower bound for $C_2(3)$ itself will be closer to $K_G^{\mathbb{C}}$.

We recall some of the details from the two papers [Var74, Var76] of Varopoulos, which will be useful in what follows. Fix a Hilbert space $\mathbb{H}$ and a bilinear form S on $\mathbb{H}$ with norm

1. Let e, f be two arbitrary but fixed vectors of length 1 and set $\mathscr{H} = \{e\} \oplus \mathbb{H} \oplus \{f\}$. For any $x \in \mathbb{H}$, define $T_x : \mathscr{H} \to \mathscr{H}$ by the rule

$$T_x f = x,\ T_x y = S(x, y)e,\ T_x e = 0 \text{ for all } y \in \mathbb{H} \tag{2.4}$$

and extend it linearly. It is then easily verified that for every $x, y \in \mathbb{H}$, T_x and T_y commute.

Lemma 2.1. *For every $x \in \mathbb{H}$, $\|T_x\| = \|x\|$, where the operator T_x is defined according to* (2.4).

Proof. For $h \in \mathscr{H}$ and $\alpha, \beta \in \mathbb{C}$, we have $h = \alpha e + P_{\mathbb{H}} h + \beta f$, where $P_{\mathbb{H}} : \mathscr{H} \to \mathscr{H}$ is the orthogonal projection on $\mathbb{H}$.

$$\langle T_x^* T_x h, h\rangle = |S(x, P_{\mathbb{H}}(h))|^2 + |\beta|^2 \|x\|^2 \leq (\|P_{\mathbb{H}} h\|^2 + |\beta|^2)\|x\|^2$$

therefore $\langle T_x^* T_x h, h\rangle \leq \|h\|^2 \|x\|^2$. Thus $\|T_x\| \leq \|x\|$. We already know that $\|T_x\| \geq \|x\|$ and hence $\|T_x\| = \|x\|$. □

Definition 2.2 (Grothendieck Constant)**.** Suppose $A := (\!(a_{jk})\!)_{n\times n}$ is a complex(real) array satisfying

$$\left|\sum_{j,k=1}^{n} a_{jk} s_j t_k\right| \leq \max\left\{\left|s_j\right| |t_k| : 1 \leq j, k \leq n\right\}, \tag{2.5}$$

where s_j, t_k are complex(real) numbers. Then there exists $K > 0$ such that for any choice of sequence of vectors $(x_j)_1^n$, $(y_k)_1^n$ in a complex(real) Hilbert space $\mathbb{H}$, we have

$$\left|\sum_{j,k=1}^{n} a_{jk} \langle x_j, y_k\rangle\right| \leq K \max\left\{\|x_j\|\|y_k\| : 1 \leq j, k \leq n\right\}. \tag{2.6}$$

The least constant K satisfying inequality (2.6) is denoted by K_G and called Grothendieck constant. The constant K_G is a universal constant independent of n and the matrices satisfying the hypothesis (2.5). Note that the definition of K_G depends only on the underlying field. When it is the field $\mathbb{C}$ of complex numbers, this constant is called the complex Grothendieck constant and is denoted by $K_G^{\mathbb{C}}$.

Let p_A be the polynomial $\sum_{i,j=1}^n a_{ij} z_i w_j$. The inequality (2.5) is equivalent to saying that $\|p_A\|_{\mathbb{D}^{2n},\infty} \leq 1$. This follows from the equality $\|p_A\|_{\mathbb{D}^{2n},\infty} = \|A\|_{\ell^\infty(n)\to\ell^1(n)}$. Let $p_{A,\triangle}$ be the restriction of p_A to the diagonal set

$$\triangle = \left\{(z_1, \ldots, z_n, z_1, \ldots, z_n) : |z_i| < 1, 1 \leq i \leq n\right\},$$

which is the polydisc $\mathbb{D}^n$. Thus $\|p_{A,\Delta}\|_{\mathbb{D}^n,\infty}$ is also at most 1. If A is symmetric, then the second derivative $D^2 p_{A,\Delta}(0)$ is $2A$. It is therefore clear that

$$\|p_{2A,\Delta}\|_{\mathbb{D}^n,\infty} \leq 2\|A\|_{\ell^\infty(n)\to\ell^1(n)}. \tag{2.7}$$

We find examples where (2.7) is strict. Indeed, for this particular example, we show that

$$\frac{\|A\|_{\ell^\infty(n)\to\ell^1(n)}}{\|p_{A,\Delta}\|_{\mathbb{D}^n,\infty}} \geq 1.2$$

This observation will be important for us in what follows. As pointed out earlier, the following theorem is due to Varopoulos.

Theorem 2.3 ([Var76]). $\lim_{n\to\infty} C(n) \geq K_G^{\mathbb{C}}$.

Proof. It is a well known that $K_G^{\mathbb{C}} > 1$. Let $\epsilon > 0$ be a fixed real number such that $K_G^{\mathbb{C}} - \epsilon > 1$. Since $K_G^{\mathbb{C}}$ is the least constant in the inequality (2.6), therefore there exists a matrix $A := (\!(a_{jk})\!)_{n\times n}$ satisfying the inequality (2.5) and unit vectors x_i, y_i, in $\ell^2(k)$, $1 \leq i \leq n$, for some $k \in \mathbb{N}$ such that

$$\sum_{j,k=1}^{n} a_{jk}\langle x_j, y_k\rangle > (K_G^{\mathbb{C}} - \epsilon)$$

Let

$$\tilde{A} = (\!(\tilde{a}_{jk})\!)_{2n\times 2n} := \frac{1}{2}\begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix}.$$

It is easy to see that $\tilde{A}$ satisfies inequality (2.5). Take $\tilde{x}_1 = x_1$, $\tilde{x}_2 = x_2, \ldots,$ $\tilde{x}_n = x_n$, $\tilde{x}_{n+1} = \bar{y}_1, \ldots,$ $\tilde{x}_{2n} = \bar{y}_n$ and consider the bilinear form S on $\ell^2(k)$ defined as follows:

$$S(x,y) = \sum_{j=1}^{k} x_j y_j,$$

where $x = (x_1,\ldots,x_k)$ and $y = (y_1,\ldots,y_k)$. The operator $\tilde{A} : \ell^\infty(2n) \to \ell^1(2n)$ is of norm at most 1 and

$$\sum_{j,k=1}^{2n} \tilde{a}_{jk} S(\tilde{x}_j, \tilde{x}_k) = \frac{1}{2}\left\{\sum_{j,k=1}^{n} a_{jk}\langle x_j, y_k\rangle + \sum_{j,k=1}^{n} a_{kj}\langle \bar{y}_j, \bar{x}_k\rangle\right\},$$

which implies

$$\sum_{j,k=1}^{2n} \tilde{a}_{jk} S(\tilde{x}_j, \tilde{x}_k) = \sum_{j,k=1}^{n} a_{jk}\langle x_j, y_k\rangle > K_G^{\mathbb{C}} - \epsilon. \tag{2.8}$$

The polynomial $p(z_1,\dots,z_{2n}) = \sum_{j,k=1}^{2n} \tilde{a}_{jk} z_j z_k$ is a homogeneous polynomial of degree two. It is clear that $\|p\|_{\mathbb{D}^{2n},\infty} \le 1$. Consider the operators (as defined in (2.4)),

$$T_{\tilde{x}_j} = \begin{pmatrix} 0 & \tilde{x}_j & 0 \\ 0 & 0 & \tilde{x}_j^t \\ 0 & 0 & 0 \end{pmatrix} \tag{2.9}$$

for $j = 1,\dots,2n$. Then $\|p(T_{\tilde{x}_1},\dots,T_{\tilde{x}_{2n}})\| > K_G^{\mathbb{C}} - \epsilon$ is a direct implication of the inequality (2.8). □

Let $\mathbb{H}$ be a separable Hilbert space and $\{e_j\}_{j\in\mathbb{N}}$ be a set of orthonormal basis for $\mathbb{H}$. For any $x \in \mathbb{H}$, define $x^\sharp : \mathbb{H} \to \mathbb{C}$ by $x^\sharp(y) = \sum_j x_j y_j$, where $x = \sum x_j e_j$ and $y = \sum y_j e_j$. For $x, y \in \mathbb{H}$, we set $\left[x^\sharp, y\right] = x^\sharp(y)$. From the definition it can be seen that $\left[x^\sharp, y\right] = \left[y^\sharp, x\right]$. Let $\mathbb{H}^\sharp := \left\{x^\sharp : x \in \mathbb{H}\right\}$. Let $\mathbb{H}^\sharp$ be equipped with the operator norm. Since the map $\phi : \mathbb{H} \to \mathbb{H}^\sharp$ defined by $\phi(x) = x^\sharp$ is a linear onto isometry, therefore $\mathbb{H}^\sharp$ is linearly (as opposed to the usual anti-linear identification) isometrically isomorphic to $\mathbb{H}$.

Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two separable Hilbert spaces and $\{e_j\}_{j\in\mathbb{N}}$, $\{\tilde{e}_j\}_{j\in\mathbb{N}}$ be orthonormal bases of $\mathbb{H}_1$ and $\mathbb{H}_2$ respectively. Let $\{f_j\}_{j\in\mathbb{N}}$ and $\{\tilde{f}_j\}_{j\in\mathbb{N}}$ be the corresponding dual basis for $\mathbb{H}_1$ and $\mathbb{H}_2$ respectively. For a linear map $T : \mathbb{H}_1 \to \mathbb{H}_2$, define $T^\sharp : \mathbb{H}_2 \to \mathbb{H}_1$ by

$$T^\sharp(\tilde{e}_k) = \sum_j \tilde{f}_k(Te_j)e_j$$

and extend it linearly. We note that if T is bounded then so is the operator $T^\sharp$. We have the following lemma:

Lemma 2.4. *Let $\mathbb{H}_1, \mathbb{H}_2$ and $\mathbb{H}_3$ be separable Hilbert spaces and $\{e_j^{(p)}\}_{j\in\mathbb{N}}$ be an orthonormal basis for $\mathbb{H}_p$ for $p = 1,2,3$. Let $\{f_j^{(p)}\}_{j\in\mathbb{N}}$ be the corresponding dual basis for $\mathbb{H}_p$ for $p = 1,2,3$. If $T : \mathbb{H}_1 \to \mathbb{H}_2$ and $S : \mathbb{H}_2 \to \mathbb{H}_3$ are two bounded operators, then $(S\circ T)^\sharp = T^\sharp \circ S^\sharp$.*

Proof. It is enough to check the equality $(S\circ T)^\sharp = T^\sharp \circ S^\sharp$ on the basis elements $\{e_k^{(3)}\}_{k\in\mathbb{N}}$. For any $k \in \mathbb{N}$,

$$\begin{aligned} (S\circ T)^\sharp\left(e_k^{(3)}\right) &= \sum_j f_k^{(3)}\left(S\left(T(e_j^{(1)})\right)\right)e_j^{(1)}. \\ \left(T^\sharp\circ S^\sharp\right)\left(e_k^{(3)}\right) &= T^\sharp\left(\sum_j f_k^{(3)}\left(Se_j^{(2)}\right)e_j^{(2)}\right) \\ &= \sum_j f_k^{(3)}\left(Se_j^{(2)}\right)\left(\sum_l f_j^{(2)}(Te_l^{(1)})e_l^{(1)}\right). \end{aligned}$$

Thus

$$
\begin{aligned}
\big(T^\sharp\circ S^\sharp\big)\big(e_k^{(3)}\big) &= \sum_l f_k^{(3)}\big(S\big(\sum_j f_j^{(2)}(Te_l^{(1)})e_j^{(2)}\big)\big)e_l^{(1)}\\
&= \sum_l f_k^{(3)}\big((S\circ T)(e_l^{(1)})\big)e_l^{(1)}.
\end{aligned}
$$

Hence $\big(S\circ T\big)^\sharp\big(e_k^{(3)}\big) = \big(T^\sharp\circ S^\sharp\big)\big(e_k^{(3)}\big)$. $\square$

The form of the operator appearing in (2.9) and the operators used in the addendum of [Var74], suggest the definition of the following two classes:

Definition 2.5 (Varopoulos Operator of Type I (V I))**.** Let $\mathbb{H}$ be a separable Hilbert space. For $x, y \in \mathbb{H}$, define $T_{x,y} : \mathbb{C}\oplus\mathbb{H}\oplus\mathbb{C} \to \mathbb{C}\oplus\mathbb{H}\oplus\mathbb{C}$ by

$$
T_{x,y} = \begin{pmatrix} 0 & x^\sharp & 0\\ 0 & 0 & y\\ 0 & 0 & 0 \end{pmatrix}.
$$

The operator $T_{x,y}$ will be called Varopoulos operator of type I corresponding to the pair of vectors x, y. If $x = y$ then T_{xy} will simply be denoted by T_x.

Definition 2.6 (Varopoulos Operator of Type II and of order k (V II of order k))**.** Let $\mathbb{H}$ be a separable Hilbert space. For $X \in \mathscr{B}(\mathbb{H})$, let

$$
T_X = \begin{pmatrix} 0 & X & 0 & \cdots & 0\\ 0 & 0 & X & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & X\\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}
$$

be the operator in $\mathscr{B}(\mathbb{H}\otimes C^{k+1})$. In analogy with the work of Varopoulos [Var76], operators of the form T_X, $X \in \mathscr{B}(\mathbb{H})$, are called Varopoulos operator of type II and of order k.

In the following section, we show that $\|p(T_1,\ldots,T_n)\| \le \|p\|_{\mathbb{D}^n,\infty}$ for any n commuting contractions of the form

$$
\left\{\left(\begin{pmatrix} \omega_1 & \alpha_1 & 0\\ 0 & \omega_1 & \beta_1\\ 0 & 0 & \omega_1\end{pmatrix},\ldots,\begin{pmatrix} \omega_n & \alpha_n & 0\\ 0 & \omega_n & \beta_n\\ 0 & 0 & \omega_n\end{pmatrix}\right) : \alpha_i\beta_j = \alpha_j\beta_i, 1\le i,j\le n, \omega := (\omega_1,\ldots,\omega_n)\in\mathbb{D}^n\right\}, \qquad (2.10)
$$

after assuming that $|\alpha_i| = |\beta_i|$, $1 \le i \le n$. This is interesting considering that the von-Neumann inequality is valid for any commuting n - tuple of 2×2 contractions [MP93, Agl90] and fails for 4×4 contractions [Hol01].

Secondly, we show that $\lim_{n\to\infty} C_2(n) \le \frac{3\sqrt{3}}{4} K_G^{\mathbb{C}}$ giving a considerable improvement on the upper bound previously obtained in [Var76].

Finally, we investigate in some detail, the contractivity of the homomorphisms $\rho_{x,y}$ induced by the Varopoulos operators of type I (V I). In particular, for a pair of commuting contractions T_{x_1}, T_{x_2}, $x_1, x_2 \in \mathbb{C}$, of type V I, and any holomorphic function $f : \mathbb{D}^2 \to \mathbb{D}$, $f(0,0) = 0$, applying the von-Neumann inequality, we must have

$$\sup_{x_1,x_2\in\mathbb{D}} \left\| \mathcal{T}\left(\frac{\partial f}{\partial z_1}(0)x_1 + \frac{\partial f}{\partial z_2}(0)x_2, \frac{1}{2}\sum_{i,j=1}^{2} \frac{\partial^2 f}{\partial z_i \partial z_j}(0)x_i x_j \right) \right\| \le 1,$$

where $\mathcal{T}(\omega,\alpha) = \begin{pmatrix} \omega & \alpha \\ 0 & \omega \end{pmatrix}$. The solution to this problem (indeed a generalization of it), which we obtain in Chapter 3, therefore gives a necessary condition for the Carathéodory-Fejér interpolation problem for polynomials of degree 2. Unfortunately, while the extremal problem can be stated for any n in $\mathbb{N}$, not just for 2, its solution depends on the commutant lifting theorem.

2.1 The von-Neumann Inequality

The von-Neumann inequality for a commuting n - tuple of 3×3 matrices remains open for $n \ge 3$. In this section, we establish this inequality for any n - tuple of commuting contractions of the form prescribed in (2.10) with the additional assumption that $|\alpha_i| = |\beta_i|$, $1 \le i \le n$.

Let $\mathbb{H}$ be a separable Hilbert space. Given a set of n operators $A_1, \ldots, A_n$ in $\mathscr{B}(\mathbb{H})$, define the operator

$$\mathcal{T}(A_1,\ldots,A_n) := \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_n \\ 0 & A_1 & A_2 & \cdots & A_{n-1} \\ 0 & 0 & A_1 & \cdots & A_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_1 \end{pmatrix},$$

which is in $\mathscr{B}(\mathbb{H} \otimes \mathbb{C}^n)$.

The condition for the contractivity of any 3×3 matrix of the form

$$T = \begin{pmatrix} \omega & \alpha & 0 \\ 0 & \omega & \beta \\ 0 & 0 & \omega \end{pmatrix}, \omega \in \mathbb{D}, \alpha \text{ and } \beta \text{ in } \mathbb{C}. \tag{2.11}$$

is given in the following lemma. It will be used repeatedly in what follows.

Lemma 2.7. *The operator T defined in (2.11) is a contraction if and only if*

$$|\alpha| \leq 1 - |\omega|^2, \ |\beta| \leq 1 - |\omega|^2$$

and

$$|\alpha\beta\omega|^2 \leq \left((1 - |\omega|^2)^2 - |\alpha|^2\right)\left((1 - |\omega|^2)^2 - |\beta|^2\right).$$

In particular, if $|\alpha| = |\beta|$, then T is contractive if and only if $|\alpha| \leq (1 - |\omega|)\sqrt{1 + |\omega|}$.

Proof. Suppose T is contraction. Then $\mathcal{T}(\omega, \alpha) := \left(\begin{smallmatrix} \omega & \alpha \\ 0 & \omega \end{smallmatrix}\right)$ and $\mathcal{T}(\omega, \beta) := \left(\begin{smallmatrix} \omega & \beta \\ 0 & \omega \end{smallmatrix}\right)$ must be contractions. Hence, we have $|\alpha| \leq 1 - |\omega|^2$ and $|\beta| \leq 1 - |\omega|^2$. By Parrott's theorem [Par78], there exists $a \in \mathbb{C}$ such that the operator T_a is a contraction, where

$$T_a = \begin{pmatrix} \omega & \alpha & a \\ 0 & \omega & \beta \\ 0 & 0 & \omega \end{pmatrix}.$$

Every possible choice of a in $\mathbb{C}$, ensuring contractivity of the operator T_a is given by

$$a = (I - ZZ^*)^{1/2} V (I - Y^* Y)^{1/2} - ZS^* Y,$$

where $V : \mathbb{C} \to \mathbb{C}$ is an arbitrary contraction, $S = \mathcal{T}(0, \omega)$, $R = (\beta, \omega)^{\mathrm{t}}$ and $Q = (\omega, \alpha)$. The operators Y and Z are explicitly determined by the formulae $R = (I - SS^*)^{1/2} Y$ and $Q = Z(I - S^* S)^{1/2}$. The reader is referred to (cf. [You88, Chapter 12]) for more information on the Parrott's theorem and related topics. Thus

$$Y = \left(\frac{\beta}{(1 - |\omega|^2)^{1/2}}, \omega\right)^{\mathrm{t}} \text{ and } Z = \left(\omega, \frac{\alpha}{(1 - |\omega|^2)^{1/2}}\right).$$

Therefore

$$a = \left((1 - |\omega|^2) - \frac{|\alpha|^2}{1 - |\omega|^2}\right)^{1/2} V \left((1 - |\omega|^2) - \frac{|\beta|^2}{1 - |\omega|^2}\right)^{1/2} - \frac{\overline{\omega}\beta\alpha}{1 - |\omega|^2}. \tag{2.12}$$

Since T is a contraction, it follows that $a=0$ is a valid choice in (2.12) for some contraction V. This forces

$$V=\frac{\overline{\omega}\beta\alpha}{1-|\omega|^2}\left((1-|\omega|^2)-\frac{|\alpha|^2}{1-|\omega|^2}\right)^{-1/2}\left((1-|\omega|^2)-\frac{|\alpha|^2}{1-|\omega|^2}\right)^{-1/2}$$

to be of absolute value at most 1. Thus we have

$$\frac{|\omega|^2|\alpha|^2|\beta|^2}{\left(1-|\omega|^2\right)^2}\le\left((1-|\omega|^2)-\frac{|\alpha|^2}{1-|\omega|^2}\right)\left((1-|\omega|^2)-\frac{|\beta|^2}{1-|\omega|^2}\right).$$

Hence we get

$$|\alpha\beta\omega|^2\le\left((1-|\omega|^2)^2-|\alpha|^2\right)\left((1-|\omega|^2)^2-|\beta|^2\right).$$

All the steps in the proof given above are reversible. Therefore, the converse statement is valid as well.

The condition for contractivity assuming $|\alpha|=|\beta|$ is easily seen to be

$$|\alpha|\le(1-|\omega|)\sqrt{1+|\omega|}.$$

$\square$

Remark 2.8. *It is known that T is a contraction if and only if $\|f(T)\|\le 1$ for all f in the disc algebra $\mathbb{A}(\mathbb{D})$ with $f(\omega)=0$ for an arbitrary but fixed ω in $\mathbb{D}$ and $\|f\|_{\mathbb{D},\infty}\le 1$. Therefore T in (2.11) is a contraction if and only if*

$$|f'(\omega)|^2|\alpha|^2+|f''(\omega)/2||\alpha|^2\le 1$$

for all $f\in\mathbb{A}(\mathbb{D})$ with $f(\omega)=0$ and $\|f\|_{\mathbb{D},\infty}\le 1$. Thus

$$|\alpha|^2\le\frac{1}{\sup\left(|f'(\omega)|^2+|\frac{f''(\omega)}{2}|\right)},$$

where the supremum is over the set $\{f\in\mathbb{A}(\mathbb{D}):f(\omega)=0,\|f\|_{\mathbb{D},\infty}\le 1\}$. From the Lemma 2.7, we conclude (for some arbitrary but fixed $\omega\in\mathbb{D}$) that

$$\sup\left\{|f'(\omega)|^2+\left|\frac{f''(\omega)}{2}\right|:f\in\mathbb{A}(\mathbb{D}),f(\omega)=0,\|f\|_{\mathbb{D},\infty}\le 1\right\}=\frac{1}{(1-|\omega|^2)(1-|\omega|)}.$$

For $\alpha_j,\beta_j\in\mathbb{D}$, define the operators

$$T_j=\begin{pmatrix}0&\alpha_j&0\\0&0&\beta_j\\0&0&0\end{pmatrix},1\le j\le n,$$

and assume that $\alpha_j\beta_k = \alpha_k\beta_j$, $j,k = 1,\dots,n$. This commuting n-tuple of contractions $T = (T_1,\dots,T_n)$ is in the set (2.10) with $\omega = 0$. It defines a homomorphism $\rho_T : \mathbb{C}[Z_1,\dots,Z_n] \to \mathscr{B}(\mathbb{C}^3)$ given by the formula $\rho_T(p) := p(T)$. Explicitly evaluating $p(T)$, we obtain

$$\rho_T(f) := \begin{pmatrix} f(0) & Df(0)\cdot\alpha & \frac{1}{2}D^2 f(0)\cdot A_{\alpha\beta} \\ 0 & f(0) & Df(0)\cdot\beta \\ 0 & 0 & f(0) \end{pmatrix}, \ f \in \mathbb{C}[Z_1,\dots,Z_n], \tag{2.13}$$

where $A_{\alpha\beta} = \left(\!\left(\alpha_j\beta_k\right)\!\right)_{n\times n}$. Here

$$Df(0)\cdot\alpha = \sum_j \frac{\partial f}{\partial z_j}(0)\alpha_j,\ Df(0)\cdot\beta = \sum_j \frac{\partial f}{\partial z_j}(0)\beta_j$$

and

$$D^2 f(0)\cdot A_{\alpha\beta} = \sum_{j,k} \frac{\partial^2 f}{\partial z_j \partial z_k}(0)\alpha_j\beta_k.$$

Clearly, the formula (2.13) makes sense and defines a homomorphism of the algebra $\mathrm{H}^\infty(\mathbb{D}^n)$ consisting of all bounded holomorphic functions on the polydisc $\mathbb{D}^n$.

The following lemma and several of its variants involving functions defined on domains in $\mathbb{C}^n$ and taking values in $k\times k$ matrices have been proved in [Mis84, MNS90, Mis94, Pau92]. The proof below follows closely the one appearing in [Pau92].

Lemma 2.9 (The zero lemma)**.** *The homomorphism ρ_T is a contraction if and only if $\|\rho_T(f)\| \le 1$ for all $f \in \mathrm{H}^\infty(\mathbb{D}^n)$ with $f(0) = 0$ and $\|f\|_{\mathbb{D}^n,\infty} \le 1$.*

Proof. Let us assume that $\|\rho_T(f)\| \le 1$ for all $f : \mathbb{D}^n \to \mathbb{D}$ with $f(0) = 0$. Let $g : \mathbb{D}^n \to \mathbb{D}$ be an analytic function and ϕ be an automorphism of $\mathbb{D}$ mapping $g(0)$ to 0. Then $\phi\circ g$ is an analytic map from $\mathbb{D}^n$ to $\mathbb{D}$ with $(\phi\circ g)(0) = 0$, therefore $\|\rho_T(\phi\circ g)\| \le 1$. Now by von-Neumann's inequality we have $\|\phi^{-1}(\rho_T(\phi\circ g))\| \le 1$ which is equivalent to $\|\rho_T(g)\| \le 1$. Hence ρ_T is a contraction. The converse is trivially true. □

Theorem 2.10. *The homomorphism ρ_T, as defined in (2.13) for the commuting tuple of contractions T, is contractive.*

Proof. Assume that the supremum norm of the polynomial

$$p(z_1,\dots,z_n) = \sum_{j=1}^n a_j z_j + \sum_{i,j=1}^n a_{ij} z_i z_j + \sum_{k=3}^d \sum_{|I|=k} a_I z^I$$

over the polydisc $\mathbb{D}^n$ is at most 1. Then

$$\|\rho_T(p)\| = \left\| \begin{pmatrix} \sum a_i\alpha_i & \sum a_{ij}\alpha_i\beta_j \\ 0 & \sum a_i\beta_i \end{pmatrix} \right\| \le 1$$

if and only if

$$\left|\sum_{i,j=1}^{n} a_{ij}\alpha_i\beta_j\right|^2 \le \left(1-\left|\sum_{j=1}^{n} a_j\alpha_j\right|^2\right)\left(1-\left|\sum_{j=1}^{n} a_j\beta_j\right|^2\right). \tag{2.14}$$

Without loss of generality we assume $0<|\beta_1|\le|\alpha_1|$. Let $|\beta_1|/|\alpha_1|=\mu$. We have $\alpha_j\beta_k = \alpha_k\beta_j$ for all $j,k=1,\ldots,n$ therefore inequality (2.14) is equivalent to

$$\left|\sum_{i,j=1}^{n} a_{ij}\alpha_i\alpha_j\right|^2 \le \left(1-\left|\sum_{j=1} a_j\alpha_j\right|^2\right)\left(\frac{1}{\mu^2}-\left|\sum_{j=1} a_j\alpha_j\right|^2\right). \tag{2.15}$$

Define $q_\alpha(t) := p(t\alpha_1,\ldots,t\alpha_n)$ for all $t\in\mathbb{D}$. Since $\|p\|_{\mathbb{D}^n,\infty}\le 1$ therefore $\|q_\alpha\|_{\mathbb{D},\infty}\le 1$ and hence $\mathcal{T}\big(\sum a_i\alpha_i, \sum a_{ij}\alpha_i\alpha_j\big)$ is of norm at most 1. Therefore

$$\left|\sum_{i,j=1}^{n} a_{ij}\alpha_i\alpha_j\right|^2 \le \left(1-\left|\sum_{j=1} a_j\alpha_j\right|^2\right)\left(1-\left|\sum_{j=1} a_j\alpha_j\right|^2\right)$$

and since $\mu\le 1$, it follows that (2.15) holds. Hence ρ_T is a contractive homomorphism.

$\square$

We now prove the von-Neumann inequality for any contractive n - tuple in the set (2.10) assuming $|\alpha_i|=|\beta_i|$, $1\le i\le n$. (We no longer assume that $\omega=0$.) As before, such a n-tuple defines a homomorphism $\rho_{T,\omega}:\mathrm{H}^\infty(\mathbb{D}^n)\to\mathscr{B}(\mathbb{C}^3)$ by

$$\rho_{T,\omega}(f) := \begin{pmatrix} f(\omega) & Df(\omega)\cdot\alpha & \frac{1}{2}D^2f(\omega)\cdot A_{\alpha\beta} \\ 0 & f(\omega) & Df(\omega)\cdot\beta \\ 0 & 0 & f(\omega)\end{pmatrix}, \tag{2.16}$$

where

$$Df(\omega)\cdot\alpha = \sum_j \frac{\partial f}{\partial z_j}(\omega)\alpha_j,\ Df(\omega)\cdot\beta = \sum_j \frac{\partial f}{\partial z_j}(\omega)\beta_j$$

and

$$D^2f(\omega)\cdot A_{\alpha\beta} = \sum_{j,k}\frac{\partial^2 f}{\partial z_j\partial z_k}(\omega)\alpha_j\beta_k.$$

Here is “the zero lemma” again, now adapted to work for the homomorphism $\rho_{T,\omega}$. For the proof, we compose f with an automorphism of the disc taking $f(\omega)$ to 0, whenever $f(\omega)\neq 0$.

Lemma 2.11. *The homomorphism $\rho_{T,\omega}$ is contractive if and only if $\|\rho_{T,\omega}(f)\|\le 1$ for all $f\in\mathrm{H}^\infty(\mathbb{D}^n)$ with $f(\omega)=0$ and $\|f\|_{\mathbb{D}^n,\infty}\le 1$.*

Theorem 2.12. *The homomorphism $\rho_{T,\omega}$ induced by a contractive n - tuple T in the set* (2.10) *with $\alpha_j = \beta_j$, $j = 1,\dots,n$, is itself contractive.*

Proof. Assume that the supremum norm of the polynomial

$$p(z_1,\dots,z_n) = \sum_{j=1}^{n} a_j(z_j-\omega_j) + \sum_{i,j=1}^{n} a_{ij}(z_j-\omega_j)(z_i-\omega_i) + \sum_{k=3}^{d}\sum_{|I|=k} a_I(z-\omega)^I$$

over the polydisc $\mathbb{D}^n$ is at most 1, where d is the degree of p. Then

$$\left\|\rho_{T,\omega}(p)\right\| = \left\|\begin{pmatrix} \sum a_i\alpha_i & \sum a_{ij}\alpha_i\alpha_j \\ 0 & \sum a_i\alpha_i \end{pmatrix}\right\| \le 1$$

if and only if

$$\left|\sum_{i,j=1}^{n} a_{ij}\alpha_i\alpha_j\right| \le \left(1-\left|\sum_{j=1}^{n} a_j\alpha_j\right|^2\right). \tag{2.17}$$

For $j = 1,\dots,n$, applying the Lemma 2.7 to the operator

$$\begin{pmatrix} \omega_j & (1-|\omega_j|)\sqrt{1+|\omega_j|} & 0 \\ 0 & \omega_j & (1-|\omega_j|)\sqrt{1+|\omega_j|} \\ 0 & 0 & \omega_j \end{pmatrix}$$

we conclude that it must be contractive. Therefore, using Nehari's theorem (cf. [You88, Chapter 15, Theorem 15.14]), we obtain a holomorphic function h_j, $h_j^{(k)}(0) = 0$, $k = 0,1,2$, defined in the unit disc $\mathbb{D}$ such that the supremum norm of the function

$$f_j(z) = \omega_j + (1-|\omega_j|)\sqrt{1+|\omega_j|}z + h_j(z)$$

over the unit disc $\mathbb{D}$ is at most 1. Define $f = (f_1,\dots,f_n) : \mathbb{D}^n \to \mathbb{D}^n$ by $f(z_1,\dots,z_n) = \big(f_1(z_1),\dots,f_n(z_n)\big)$. Now $p\circ f$ maps $\mathbb{D}^n$ to $\mathbb{D}$ with $p\circ f(0) = 0$. Define the following contractive operators

$$S_j := \begin{pmatrix} 0 & \frac{\alpha_j}{(1-|\omega_j|)\sqrt{1+|\omega_j|}} & 0 \\ 0 & 0 & \frac{\alpha_j}{(1-|\omega_j|)\sqrt{1+|\omega_j|}} \\ 0 & 0 & 0 \end{pmatrix}$$

for $j = 1,\dots,n$. Then $S = (S_1,\dots,S_n)$ is a tuple of commuting contractions. From Theorem 2.10 it is clear that $\|p\circ f(S)\| \le 1$. Therefore (2.17) holds and hence $\|p(T_1,\dots,T_n)\| \le 1$. $\square$

As a corollary of this theorem, we get the following necessary condition for the Carathéodory-Fejér interpolation problem for the polydisc $\mathbb{D}^n$.

Theorem 2.13. *Let p be a polynomial in n variables of degree 2 such that $p(0) = 0$. There exists a holomorphic function q, defined on polydisc $\mathbb{D}^n$, with $q^{(k)}(0) = 0$, $|k| \le d$ such that $\|p + q\|_\infty \le 1$ only if*

$$\sup_{\|\alpha\|_\infty \le 1} \left\{ \left| \frac{D^2 p(0) \cdot A_\alpha}{2} \right| + \left| Dp(0) \cdot \alpha \right|^2 \right\} \le 1,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $A_\alpha = \left(\!\left(\alpha_i \alpha_j\right)\!\right)$, $D^2 p(0) \cdot A_\alpha = \sum \frac{\partial^2 p}{\partial z_i \partial z_j}(0) \alpha_i \alpha_j$ and $Dp(0) \cdot \alpha = \sum \frac{\partial p}{\partial z_i}(0) \alpha_i$.

Remark 2.14. *This proof of the von-Neumann inequality works without having to make the assumption that $|\alpha_i| = |\beta_i|$, if instead, we assume that*

$$\begin{pmatrix} \omega_j & \alpha_j & 0 \\ 0 & \omega_j & \alpha_j \\ 0 & 0 & \omega_j \end{pmatrix} \text{ and } \begin{pmatrix} \omega_j & \beta_j & 0 \\ 0 & \omega_j & \beta_j \\ 0 & 0 & \omega_j \end{pmatrix}$$

are contractions for $j = 1, \ldots, n$. Unfortunately, there are contractive n - tuples T in the set (2.10) for which this condition is not met, for example, take $\omega = \alpha = 2/5$ and $\beta = 4/5$, here n is just 1!

2.2 An improvement in the bound for $C_2(n)$

The explicit example in [Var74] showing that a commuting triple of contractions need not define a contractive homomorphism of the tri-disc algebra uses the interesting polynomial

$$p_V(z_1, z_2, z_3) := z_1^2 + z_2^2 + z_3^2 - 2z_1 z_2 - 2z_2 z_3 - 2z_3 z_1.$$

This polynomial will be referred as the Varopoulos-Kaijser polynomial. The supremum norm of p_V over the tri-disc is shown to be 5. Let

$$A_V := \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \tag{2.18}$$

be the matrix of co-efficients of the polynomial p_V.

Lemma 2.15. $\|A_V\|_{\ell^\infty(3) \to \ell^1(3)} \ge 6$.

Proof. Suppose a_{ij} denote the (i,j) entry of A_V and $z_j = e^{i\theta_j}$ for $i,j = 1,2,3$. Then

$$\begin{aligned}
\left|\sum a_{ij} z_i \overline{z}_j\right| &= |z_1|^2 + |z_2|^2 + |z_3|^2 - 2Re\left(z_1\overline{z}_2 + z_2\overline{z}_3 + z_3\overline{z}_1\right) \\
&= 3 - 2(cos(\theta_1 - \theta_2) + cos(\theta_2 - \theta_3) + cos(\theta_3 - \theta_1)) \\
&\leq 3 - 2\left(\frac{-3}{2}\right) \\
&= 6.
\end{aligned}$$

Using the Lemma 2.18, the above inequality can easily be deduced . For $\theta_1 - \theta_2 = \frac{2\pi}{3} = \theta_2 - \theta_3$, the inequality in this computation becomes an equality. Thus

$$\|A_V\|_{\ell^\infty(3)\to\ell^1(3)} \geq \sup_{|z_j|=1} \left|\sum a_{ij} z_i \overline{z}_j\right| = 6.$$

□

Thus $\|A_V\|_{\ell^\infty(3)\to\ell^1(3)} > \|p_V\|_{\mathbb{D}^3,\infty}$. Here $\frac{\|A_V\|_{\ell^\infty(3)\to\ell^1(3)}}{\|p_V\|_{\mathbb{D}^3,\infty}} \geq 1.2$.

Question: Does there exists $k > 0$ such that $\|A\|_{\ell^\infty(n)\to\ell^1(n)} \leq k\|p_{A,\Delta}\|_{\mathbb{D}^n,\infty}$ for all symmetric matrices A of size n, $n \in \mathbb{N}$?

We have just seen that k is bounded below by 1.2. Now, we show that $\frac{3\sqrt{3}}{4}$ is an upper bound for k. This will be an immediate corollary of the following theorem giving an upper bound for the second derivative.

Theorem 2.16. *If $f : \mathbb{D}^n \to \mathbb{D}$ is a holomorphic function, then $\|D^2 f(0)\|_{\ell^\infty(n)\to\ell^1(n)}$ is bounded above by $\frac{3\sqrt{3}}{2}$.*

Proof. Let f be a complex valued analytic function on $\mathbb{D}^n$ with $\|f\|_{\mathbb{D}^n,\infty} \leq 1$. Let $a = (a_1,\ldots,a_n) \in \mathbb{D}^n$ be an arbitrary point. Let Φ_j be the automorphism of the unit disc defined by

$$\Phi_j(z) = \frac{z + a_j}{1 + \overline{a}_j z}$$

for $j = 1,\ldots,n$. Let $\Phi(z_1,\ldots,z_n) = (\Phi_1(z_1),\ldots,\Phi_n(z_n))$ and φ be the automorphism of the unit disc such that $\varphi(f(a)) = 0$. Due to chain rule we have

$$D(\varphi \circ f \circ \Phi)(0) = \varphi'(f(a))Df(a)D\Phi(0).$$

As $g := \varphi \circ f \circ \Phi : \mathbb{D}^n \to \mathbb{D}$ is an analytic map therefore due to Schwarz's lemma $Dg(0)$ is a

contractive linear functional on $(\mathbb{C}^n, \|\cdot\|_{\mathbb{D}^n,\infty})$. Also

$$D\Phi(0) = \begin{pmatrix} 1-|a_1|^2 & 0 & \cdots & 0 \\ 0 & 1-|a_2|^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-|a_n|^2 \end{pmatrix}$$

therefore

$$Df(a) = \varphi'(f(a))^{-1}\left(\sum_{j=1}^{n} \frac{\partial_j g(0)}{1-|a_j|^2}\right).$$

Thus we have

$$\left\|Df(a)\right\|_1 \le (1-|f(a)|^2)\max_j \frac{1}{1-|a_j|^2}.$$

Suppose $r \in (0,1)$ is such that $|a_i| < r$ for all $i = 1, \ldots, n$. Then we have

$$\|Df(a)\|_1 \le \frac{1}{1-r^2}. \tag{2.19}$$

Let $g := Df$ then g is a map from $r\mathbb{D}^n$ to $\frac{1}{1-r^2}(\mathbb{D}^n)^*$ where $(\mathbb{D}^n)^*$ denotes the dual unit ball of $(\mathbb{C}^n, \|\cdot\|_{\mathbb{D}^n,\infty})$. Now due to Schwarz's lemma $Dg(0)$ is a linear operator on $\mathbb{C}^n$ which maps $r\mathbb{D}^n$ into $\frac{1}{1-r^2}(\mathbb{D}^n)^*$. Hence we have

$$\|D^2 f(0)\|_{\ell^\infty(n)\to\ell^1(n)} \le \frac{1}{r(1-r^2)}. \tag{2.20}$$

Inequality (2.20) is true for every $r \in (0,1)$ and maximum of $r(1-r^2)$ is attained at $r = 1/\sqrt{3}$. Therefore we can conclude that

$$\|D^2 f(0)\|_{\ell^\infty(n)\to\ell^1(n)} \le \frac{3\sqrt{3}}{2}.$$

□

Let $p(z_1, z_2, \ldots, z_n) = \sum a_{ij} z_i z_j$ be a homogeneous polynomial of degree 2 in n variables with $\|p\|_{\mathbb{D}^n,\infty} \le 1$. As $D^2 p(0) = (\!(2a_{ij})\!)_{n\times n}$ therefore from (2.16), we have

$$\|(a_{ij})\|_{\ell^\infty(n)\to\ell^1(n)} \le \frac{3\sqrt{3}}{4} \approx 1.3. \tag{2.21}$$

This leads to a considerable improvement in one of the theorems of [Var76], which is exactly the same as the theorem below except that the constant obtained in [Var76] is $2K_G^{\mathbb{C}}$.

Theorem 2.17. *Suppose p be a polynomial of degree atmost 2 in n variables and $T = (T_1,\dots,T_n)$ be a tuple of commuting contractions on a Hilbert space $\mathbb{H}$. Then*

$$\|p(T_1,\dots,T_n)\| \le \frac{3\sqrt{3}}{4} K_G^{\mathbb{C}} \|p\|_{\mathbb{D}^n,\infty},$$

where $K_G^{\mathbb{C}}$ is the complex Grothendieck constant.

Proof. Let

$$p(z_1,\dots,z_n) = a_0 + \sum_{j=1}^{n} a_j z_j + \sum_{j,k=1}^{n} a_{jk} z_j z_k.$$

For $x, y \in \mathbb{H}$ arbitrary vectors of norm at most 1, we have

$$\left|\langle p(T_1,\dots,T_n)x, y\rangle\right| = \left|a_0\langle x, y\rangle + \sum_{j=1}^{n}\langle a_j T_j x, y\rangle + \sum_{j,k=1}^{n}\langle a_{jk} T_j x, T_k^* y\rangle\right|.$$

Let

$$B = \begin{pmatrix} a_0 & a_1/2 & a_2/2 & \cdots & a_n/2 \\ a_1/2 & a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_n/2 & a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

and q be the corresponding homogeneous polynomial of degree 2 in $n+1$ variables defined by

$$q(z_0, z_1,\dots,z_n) = a_0 z_0^2 + \sum_{j=1}^{n} a_j z_j z_0 + \sum_{j,k=1}^{n} a_{jk} z_j z_k.$$

It can easily be seen that $\|q\|_{\mathbb{D}^{n+1},\infty} = \|p\|_{\mathbb{D}^n,\infty}$. Suppose $v_0 = x, v_j = T_j x$ and $w_0 = y, w_j = T_j^* y$ for $j = 1,\dots,n$. Then

$$\sum_{j=0}^{n} b_{jk}\langle v_j, w_k\rangle = a_0\langle x, y\rangle + \sum_{j=1}^{n}\langle a_j T_j x, y\rangle + \sum_{j,k=1}^{n}\langle a_{jk} T_j x, T_k^* y\rangle,$$

where b_{jk} is the (j,k) entry in B. Now from the definition of the complex Grothendieck constant, we get

$$\left|\sum_{j=0}^{n} b_{jk}\langle v_j, w_k\rangle\right| \le K_G^{\mathbb{C}} \|B\|_{\ell^\infty(n+1)\to\ell^1(n+1)}.$$

Now, to complete the proof, one merely has to apply the inequality (2.21). □

2.3 Homomorphisms induced by operators of type $\mathbb{VI}$

Let Ω be a bounded domain in $\mathbb{C}^m$ and $\omega = (\omega_1, ..., \omega_m) \in \Omega$ be fixed. Let $\mathbb{H}$ be a separable Hilbert space. Let $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m)$, where $x_j, y_j \in \mathbb{H}$ for all $j = 1, \ldots, m$, be such that $[x_j^\sharp, y_k] = [x_k^\sharp, y_j]$ for all $j, k = 1, \ldots, m$. Let the operator T_{x_j,y_j} be of type $\mathbb{VI}$ corresponding to the pair x_j, y_j, $j = 1, \ldots, m$. We let $\boldsymbol{T}_{x,y}^{(\omega)}$ denote the commuting n-tuple $(\omega_1 I + T_{x_1,y_1}, \ldots, \omega_m I + T_{x_m,y_m})$. We will let $\boldsymbol{T}_x$ denote the m-tuple $(T_{x_1,x_1}, \ldots, T_{x_m,x_m})$. It is easy to see that for $j, k, l = 1, \ldots, m$, we have $T_{x_j,y_j} T_{x_k,y_k} T_{x_l,y_l} = 0$ and

$$T_{x_j,y_j} T_{x_k,y_k} = \begin{pmatrix} 0 & 0 & [x_j^\sharp, y_k] \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consequently, for any polynomial p in m variables, we see that

$$p(\boldsymbol{T}_{x,y}^{(\omega)}) = \begin{pmatrix} p(\omega) & Dp(\omega)\cdot x^\sharp & \frac{1}{2}D^2 p(\omega)\cdot A_{x,y} \\ 0 & p(\omega) I & Dp(\omega)\cdot y \\ 0 & 0 & p(\omega) \end{pmatrix}, \tag{2.22}$$

where $x^\sharp = \left(x_1^\sharp, \ldots, x_m^\sharp\right)$, $A_{x,y} = \left(\!\left([x_i^\sharp, y_j]\right)\!\right)_{m\times m}$. Therefore, extending this definition to functions in $\mathrm{H}^\infty(\Omega)$, we obtain the homomorphism $\rho_{x,y}^{(\omega)} : \mathrm{H}^\infty(\Omega) \to \mathscr{B}(\mathbb{C} \oplus \mathbb{H} \oplus \mathbb{C})$, which for any polynomial p is given by the formula $\rho_{x,y}^{(\omega)}(p) = p(\boldsymbol{T}_{x,y}^{(\omega)})$ and is defined for f in $\mathrm{H}^\infty(\Omega)$ by the same formula. The homomorphism $\rho_{x,x}^{(0)}$ will simply be denoted by ρ_x.

Suppose $\Omega = \mathbb{D}^m$ and $\|x_j\| \le 1, \|y_j\| \le 1$ for each $j = 1, \ldots, m$. Then for $m = 1, 2$, we know that $\rho_{x,y}^{(\omega)}$ is contractive homomorphism. What about $m > 2$?
The following example is due to Varopoulos and Kaijser in [Var74]. Set

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} & 0 \end{pmatrix}$$

and

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} & 0 \end{pmatrix}.$$

It is easy to see that A_1, A_2 and A_3 are commuting contractions. Now, consider the Varopoulos-Kaijser polynomial p_V defined earlier. Choose $x_1 = (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, $x_2 = (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, $x_3 = (-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and $y_1 = (1,0,0)$, $y_2 = (0,1,0)$, $y_3 = (0,0,1)$. In the notations above $T_{x_1,y_1} = A_1$, $T_{x_2,y_2} = A_2$ and $T_{x_3,y_3} = A_3$. We have

$$\|p_V(T_{x_1,y_1}, T_{x_2,y_2}, T_{x_3,y_3})\| = \Big|\sum_{j,k=1}^{3} a_{jk}\left[x_j^\sharp, y_k\right]\Big| = 3\sqrt{3} > 5 = \|p_V\|_{\mathbb{D}^3,\infty},$$

where $(\!(a_{jk})\!) = A_V$. Hence $\rho^0_{x,y}$ corresponding to $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ is not contractive. In this example the ratio of $\|p_V(\boldsymbol{T}^0_{x,y})\|$ to $\|p_V\|_{\mathbb{D}^3,\infty}$ is approximately 1.04. In this section we shall show that

$$\sup\left\{\frac{\|p_V(\boldsymbol{T}_x)\|}{\|p_V\|_{\mathbb{D}^3,\infty}} : \|x\|_2 = 1\right\}$$

is 1.2, which was proved earlier by Holbrook [Hol01]. However, we give many examples of operators of type VI for which this upper bound is attained. As explained earlier, the hope that we may be able to increase it even further was the motivation behind introducing the set of operators VI. For the proof, we shall need the following lemma.

Lemma 2.18. *For $n > 1$, we have*

$$\min\big(\langle x_1, x_2\rangle + \langle x_2, x_3\rangle + \langle x_3, x_1\rangle\big) = -\frac{3}{2},$$

where the minimum is over the set $\{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}^n,\ \|x_i\|_2 = 1,\ i = 1,2,3\}$.

Proof. Let $x_1, x_2, x_3 \in \mathbb{R}^n$ with $\|x_i\|_2 = 1$, $i = 1,2,3$. The following identity is easily verified:

$$\|x_1 + x_2 + x_3\|_2^2 = \|x_1\|_2^2 + \|x_2\|_2^2 + \|x_3\|_2^2 + 2\big(\langle x_1, x_2\rangle + \langle x_2, x_3\rangle + \langle x_3, x_1\rangle\big).$$

For $i = 1,2,3$, $\|x_i\|_2 = 1$, therefore

$$\|x_1 + x_2 + x_3\|_2^2 - 3 = 2\big(\langle x_1, x_2\rangle + \langle x_2, x_3\rangle + \langle x_3, x_1\rangle\big).$$

Thus $\langle x_1, x_2\rangle + \langle x_2, x_3\rangle + \langle x_3, x_1\rangle$ is minimized at $x_1, x_2, x_3 \in \mathbb{R}^n$ such that $x_1 + x_2 + x_3 = 0$. Choose any three points x_1, x_2, x_3 from the unit sphere of $\mathbb{R}^n$ such that the centroid of these points is the origin. For example, choose $x_1 = (1, 0, \ldots, 0)$, $x_2 = (-1/2, \sqrt{3}/2, 0, \ldots, 0)$ and $x_3 = (-1/2, -\sqrt{3}/2, 0, \ldots, 0)$. Thus we have proved the lemma. $\square$

What follows is an easy generalization of the preceding lemma.

Lemma 2.19. *For $n > 1$, we have*

$$\min\left(\sum_{i<j}\langle x_i, x_j\rangle\right) = -\frac{m}{2},$$

where minimum is over the set $\{(x_1,\dots,x_m) : x_1,\dots,x_m \in \mathbb{R}^n,\ \|x_i\|_2 = 1,\ i = 1,\dots,m\}$.

Let $x_1, x_2, x_3 \in \mathbb{R}^n$ be arbitrary vectors of Euclidean norm 1 and set $x = (x_1, x_2, x_3)$. Consider the algebra homomorphism ρ_x as in (2.22), namely, $\rho_x(p) = p(\boldsymbol{T}_x)$. Take Varopoulos-Kaijser polynomial p_V. By the definition of ρ_x, it is easy to see that

$$\begin{aligned}
\|\rho_x(p_V)\| &= \Big|\sum_{j,k=1}^{3} a_{jk}[x_j^\sharp, x_k]\Big| = \Big|\sum_{j,k=1}^{3} a_{jk}\langle x_j, x_k\rangle\Big| \\
&= \sum_{i=1}^{3} a_{ii} + 2a_{12}\langle x_1, x_2\rangle + 2a_{23}\langle x_2, x_3\rangle + 2a_{31}\langle x_3, x_1\rangle \\
&= 3 - 2\left(\langle x_1, x_2\rangle + \langle x_2, x_3\rangle + \langle x_3, x_1\rangle\right).
\end{aligned}$$

From the Lemma 2.18, it is clear that we can choose $x_1, x_2, x_3 \in \mathbb{R}^n$ (in fact there are infinitely many choices for x for each $n > 1$) such that $\|\rho_x(p_V)\| = 6$ and $\|x_i\|_2 = 1$ for each $i = 1,2,3$. Thus

$$\frac{\|\rho_x(p_V)\|}{\|p_V\|_{\mathbb{D}^3,\infty}} = \frac{6}{5} = 1.2 > 1.$$

Hence for this choice of x the corresponding ratio of $\|p_V(\boldsymbol{T}_x)\|$ to $\|p_V\|_{\mathbb{D}^3,\infty}$ is 1.2.

We state "the zero lemma" for a third time, in the form we will use it here. The proof is no different from what has been indicated earlier.

Lemma 2.20. *For $m-$tuple of vectors, $x = (x_1,\dots,x_m), y = (y_1,\dots,y_m)$ from $\mathbb{H}$, we have, $\|\rho_{x,y}^{(\omega)}(f)\| \le 1$ for all $f \in \mathrm{H}^\infty(\Omega,\mathbb{D})$ if and only if $\|\rho_{x,y}^{(\omega)}(f)\| \le 1$ for all $f \in \mathrm{H}^\infty_\omega(\Omega,\mathbb{D})$.*

As before, using the Lemma 2.20 we may assume that $f(\omega) = 0$, without loss of generality, in determining the contractivity of $\rho_{x,y}^{(\omega)}$ for f in any algebra of holomorphic functions containing the algebra $\mathrm{H}^\infty(\Omega)$.

Proposition 2.21. *Let $\rho_{x,y}^{(\omega)}$ be as defined in (2.22). For $f \in \mathrm{H}^\infty_\omega(\Omega,\mathbb{D})$, we get $\|\rho_{x,y}^{(\omega)}(f)\| \le 1$ if and only if*

$$\left|\frac{1}{2}D^2 f(\omega)\cdot A_{x,y}\right|^2 \le \left(1 - \|Df(\omega)\cdot x\|^2\right)\left(1 - \|Df(\omega)\cdot y\|^2\right).$$

Proof. Let $f \in \mathrm{H}^\infty_\omega(\Omega,\mathbb{D})$. Let $V_1 : \mathbb{H} \to \ell^2$ and $V_2 : \mathbb{H}^\sharp \to (\ell^2)^\sharp$ be isometries taking $Df(\omega)\cdot y$ to $\|Df(\omega)\cdot y\|e_1$ and $Df(\omega)\cdot x^\sharp$ to $\|Df(\omega)\cdot x\|e_1^{\mathrm{t}}$ respectively, where e_1 is $(1,0,0,\ldots)^{\mathrm{t}}$. Then

$$\begin{aligned}\|\rho^{(\omega)}_{x,y}(f)\| &= \left\|\begin{pmatrix} Df(\omega)\cdot x^\sharp & \frac{1}{2}D^2 f(\omega)\cdot A_{x,y} \\ 0 & Df(\omega)\cdot y \end{pmatrix}\right\| \\ &= \left\|\begin{pmatrix} \frac{1}{2}D^2 f(\omega)\cdot A_{x,y} & Df(\omega)\cdot x^\sharp \\ Df(\omega)\cdot y & 0 \end{pmatrix}\right\|.\end{aligned}$$

As norms are preserved under isometries therefore

$$\|\rho^{(\omega)}_{x,y}(f)\| = \left\|\begin{pmatrix} 1 & 0 \\ 0 & V_1 \end{pmatrix}\begin{pmatrix} \frac{1}{2}D^2 f(\omega)\cdot A_{x,y} & Df(\omega)\cdot x^\sharp \\ Df(\omega)\cdot y & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & V_2 \end{pmatrix}\right\|,$$

and hence

$$\begin{aligned}\|\rho^{(\omega)}_{x,y}(f)\| &= \left\|\begin{pmatrix} \frac{1}{2}D^2 f(\omega)\cdot A_{x,y} & \|Df(\omega)\cdot x\|e_1^{\mathrm{t}} \\ \|Df(\omega)\cdot y\|e_1 & 0 \end{pmatrix}\right\| \\ &= \left\|\begin{pmatrix} \frac{1}{2}D^2 f(\omega)\cdot A_{x,y} & \|Df(\omega)\cdot x\| \\ \|Df(\omega)\cdot y\| & 0 \end{pmatrix}\right\|.\end{aligned}$$

Thus we have the proposition.

□

Let
$$\mathscr{D}^{(\omega)}_\Omega := \left\{\left(\frac{1}{2}D^2 f(\omega), Df(\omega)\right) \mid f \in \mathrm{H}^\infty_\omega(\Omega,\mathbb{D})\right\}$$
be a subset of $M^s_m \times \mathbb{C}^m$, where M^s_m denotes the set of all $m\times m$ complex symmetric matrices.

Lemma 2.22. *The set $\mathscr{D}^{(\omega)}_\Omega$ can be realized as the unit ball in $M^s_m \times \mathbb{C}^m$ with respect to some norm, say $\|\cdot\|_{\mathscr{D}}$.*

Proof. We will show that $\mathscr{D}^{(\omega)}_\Omega$ is a balanced, convex and absorbing subset of $M^s_m \times \mathbb{C}^m$.

- **Balanced:** If $\lambda \in \mathbb{D}$ and $\left(\frac{1}{2}D^2 f(\omega), Df(\omega)\right) \in \mathscr{D}^{(\omega)}_\Omega$, then

  $$\lambda\left(\frac{1}{2}D^2 f(\omega), Df(\omega)\right) = \left(\frac{1}{2}D^2(\lambda f)(\omega), D(\lambda f)(\omega)\right).$$

  The map $\lambda f : \Omega \to \mathbb{D}$ is analytic with $\lambda f(\omega) = 0$ and hence

  $$\lambda\left(\frac{1}{2}D^2 f(\omega), Df(\omega)\right) \in \mathscr{D}^{(\omega)}_\Omega.$$

- **Convex:** Pick
$$\left(\frac{1}{2}D^2 f(\omega), Df(\omega)\right), \left(\frac{1}{2}D^2 g(\omega), Dg(\omega)\right) \in \mathscr{D}_\Omega^{(\omega)}.$$
For the $h := tf + (1-t)g,\ t \in (0,1)$, we have
$$\begin{aligned} &t\left(\frac{1}{2}D^2 f(\omega), Df(\omega)\right) + (1-t)\left(\frac{1}{2}D^2 g(\omega), Dg(\omega)\right) \\ &= \left(\frac{1}{2}D^2 h(\omega), Dh(\omega)\right). \end{aligned}$$
Since f, g are in $\mathrm{H}^\infty_\omega(\Omega, \mathbb{D})$, it follows that h is also in $\mathrm{H}^\infty_\omega(\Omega, \mathbb{D})$. Hence
$$t\left(\frac{1}{2}D^2 f(\omega), Df(\omega)\right) + (1-t)\left(\frac{1}{2}D^2 g(\omega), Dg(\omega)\right) \in \mathscr{D}_\Omega^{(\omega)}.$$

- **Absorbing:** Let $B = (b_{jk})$ be a symmetric matrix of order m and $a = (a_1, \ldots, a_m)$ in $\mathbb{C}^m$. Define
$$p(z_1, z_2, \ldots, z_m) = \sum_{j=1}^m a_j(z_j - \omega_j) + \sum_{j,k=1}^m b_{jk}(z_j - \omega_j)(z_k - \omega_k).$$
The function
$$f(z_1, z_2, \ldots, z_m) = \frac{p(z_1, z_2, \ldots, z_m)}{\|p\|_{\Omega,\infty}}.$$
is clearly in $\mathrm{H}^\infty_\omega(\Omega, \mathbb{D})$ with
$$Df(\omega) = \frac{a}{\|p\|_{\Omega,\infty}} \text{ and } \frac{1}{2}D^2 f(\omega) = \frac{B}{\|p\|_{\Omega,\infty}}.$$
Hence
$$\frac{1}{\|p\|_{\Omega,\infty}}(B, a) \in \mathscr{D}_\Omega^{(\omega)}.$$

□

The set
$$\mathbb{U} := \left\{(z, v_1, v_2) : z \in \mathbb{C}, v_1, v_2 \in \mathbb{H} \text{ with } |z|^2 \le \left(1 - \|v_1\|^2\right)\left(1 - \|v_2\|^2\right)\right\}$$
is seen to be the unit ball via the identification $(z, v_1, v_2) \to \left(\begin{smallmatrix} v_1^\sharp & z \\ 0 & v_2 \end{smallmatrix}\right)$. Clearly, $\|\left(\begin{smallmatrix} v_1^\sharp & z \\ 0 & v_2 \end{smallmatrix}\right)\| \le 1$ if and only if (z, v_1, v_2) is in $\mathbb{U}$. Thus we have proved the following lemma.

Lemma 2.23. *The set $\mathbb{U}$ is the unit ball with respect to the norm $\|(z, v_1, v_2)\|_{\mathbb{U}} := \|\left(\begin{smallmatrix} v_1^\sharp & z \\ 0 & v_2 \end{smallmatrix}\right)\|$.*

For fixed $x=(x_1,\dots,x_m)$, $y=(y_1,\dots,y_m)$ in $\mathbb{H}^m$, define a linear map $L^{(\omega)}_{x,y}: M^s_m \times \mathbb{C}^m \to \mathbb{C}\oplus\mathbb{H}\oplus\mathbb{H}$ by the formula

$$L^{(\omega)}_{x,y}(B,a)=\left(\frac{1}{2}\mathrm{tr}(A_{x,y}B), a\cdot x, a\cdot y\right),$$

where $a\cdot x = a_1x_1+\cdots+a_mx_m$, $a=(a_1,\dots,a_m)\in\mathbb{C}^m$ (and $a\cdot y$ is defined similarly). Proposition 2.21 together with what we have said here amounts to the equivalence asserted in the following theorem.

Theorem 2.24. *The following statements are equivalent:*

1. $\rho^{(\omega)}_{x,y}$ *is a contractive homomorphism.*
2. $L^{(\omega)}_{x,y}:(M^s_m\times\mathbb{C}^m, \|.\|_{\mathscr{D}})\to(\mathbb{C}\oplus\mathbb{H}\oplus\mathbb{H}, \|.\|_{\mathbb{U}})$ *is a contractive linear map.*

Let E be a domain (containing 0) in $\mathbb{C}$. For each $k\in\mathbb{N}_0$, let

$$\mathscr{P}_k(\Omega,E)=\left\{p\in\mathbb{C}[Z_1,\dots,Z_m]:\deg(p)\le k \text{ and } p(\Omega)\subset E\right\}.$$

For each $\omega\in\Omega$, let $\mathscr{P}^{\omega}_k(\Omega,E)$ denote the set of all polynomials $p\in\mathscr{P}_k(\Omega,E)$ such that $p(\omega)=0$.

Now, suppose Ω is the unit disc and $\omega=0$. Then we have the following theorem.

Theorem 2.25. *If $\mathbb{H}$ is a separable Hilbert space and $x\in\mathbb{H}$ with $\|x\|\le 1$, then for the homomorphism ρ_x, we have*

$$\sup\left\{\|\rho_x(p)\|: p\in\mathscr{P}^0_1(\mathbb{D},\mathbb{D})\right\}=\sup\left\{\|\rho_x(f)\|: f\in \mathrm{H}^\infty_0(\mathbb{D},\mathbb{D})\right\}.$$

Proof. We know that for $f\in\mathrm{H}^\infty(\mathbb{D})$ with $f(0)=0$,

$$\|\rho_x(f)\|=\left\|\begin{pmatrix}\frac{1}{2}f''(0)[x^\sharp,x] & \|f'(0)x\|\\ \|f'(0)x\| & 0\end{pmatrix}\right\|,$$

therefore from the formula in [MNS90],

$$\|\rho_x\|=\left\{|a|\|x\|^2+\frac{1}{2}\left[|a|^2\left|[x^\sharp,x]\right|^2+\sqrt{|b|^4\left|[x^\sharp,x]\right|^4+4|a|^2\|x\|^2|b|^2\left|[x^\sharp,x]\right|^2}\right]\right\}^{\frac{1}{2}},$$

where $a=f'(0)$ and $b=f''(0)/2$. Using Cauchy-Schwarz inequality we get

$$\|\rho_x\|\le\left\{|a|\|x\|^2+\frac{1}{2}\left[|a|^2\|x\|^4+\sqrt{|b|^4\|x\|^8+4|a|^2|b|^2\|x\|^6}\right]\right\}^{\frac{1}{2}}$$

and therefore

$$\|\rho_x\| \le \|x\| \left\{ |a| + \frac{1}{2}\left[|a|^2 + \sqrt{|b|^4 + 4|a|^2|b|^2} \right] \right\}^{\frac{1}{2}}.$$

Hence $\sup\left\{\|\rho_x(f)\| : f \in \mathrm{H}_0^\infty(\mathbb{D},\mathbb{D})\right\} = \|x\|$. It is easy to see that

$$\sup\left\{\|\rho_x(p)\| : p \in \mathscr{P}_1^0(\mathbb{D},\mathbb{D})\right\} = \|x\|.$$

Hence the proof is complete. □

The following corollary is now evident.

Corollary 2.26. *Suppose* $\mathbb{H}$ *is a separable Hilbert space and* $x \in \mathbb{H}$ *with* $\|x\| = 1$. *Then for homomorphism* ρ_x *defined above, we get*

$$\sup\left\{\|\rho_x(p)\| : p \in \mathscr{P}_1^0(\mathbb{D},\mathbb{D})\right\} = \sup\left\{\|\rho_x(f)\| : f \in \mathrm{H}^\infty(\mathbb{D},\mathbb{D})\right\}.$$

2.4 The Carathéodory-Fejér Interpolation Problems

We state the well known interpolation problem in m variables, usually known as the Carathéodory-Fejér (CF) problem.

Problem 2.27 (CF)**.** Given any polynomial p in m variables of degree d, find necessary and sufficient conditions on the co-efficients of p to ensure the existence of a holomorphic function h defined on the polydsic $\mathbb{D}^m$ with $h^{(k)}(0) = 0$ for all multi indices k of length at most d, such that $f := p + h$ maps the polydisc $\mathbb{D}^m$ to the unit disc $\mathbb{D}$.

Without loss of generality one may assume that $p(0) = 0$ via the transitivity of the unit disc $\mathbb{D}$. There are several different known solutions to the CF problem when $n = 1$, see (cf. [Nik86, Page 179]). However, repeated attempts to obtain solutions for $n > 1$ has remained unsuccessful for the most part, however see (cf. [BW11, Chapter 3]) for a comprehensive survey of recent results. In these notes we shall obtain necessary condition for the CF problem for the bi-disc $\mathbb{D}^2$. (However, we first discuss the case of the unit disc $\mathbb{D}$, which paves the way for the case of the bi-disc $\mathbb{D}^2$.) We show that for certain class of polynomials of degree at most 2, our necessary conditions turn out to be sufficient as well. None the less, they are not always sufficient as we demonstrate by means of an example.

We point out that the necessary condition for the CF problem actually works for any n, via an adaptation of a theorem due to Korányi and Pukánszky [KP63]. However for $n > 2$, the computations involved in deriving the necessary condition explicitly is cumbersome. Therefore, we don't give the details except in the case $n = 2$.

2.4.1 CF problem in one variable

The CF problem for one variable is stated below for polynomials of degree at most two and with $p(0) = 0$. This is the first non-trivial case of the CF problem and is typical of all other cases.

Problem 2.28. Fix p to be the polynomial $p(z) = az+bz^2$. Find a necessary and sufficient condition for the existence of a holomorphic function g defined on the unit disc $\mathbb{D}$ with $g^{(k)}(0) = 0$, $k = 0, 1, 2$, such that $\|p + g\|_{\mathbb{D},\infty} \leq 1$.

Let T_x be an operator of the type VI for some $x \in \mathbb{C}$. For any $f \in H_0^\infty(\mathbb{D},\mathbb{D})$, picking $|x| \leq 1$ to ensure contractivity of T_x, we see that $\|\rho_x(f)\| \leq 1$. Now, applying Corollary 2.21, we find that

$$\left|\frac{1}{2}f''(0)x^2\right| + \left|f'(0)x\right|^2 \leq 1.$$

Taking supremum over all x such that $|x| \leq 1$, we get

$$\left|\frac{1}{2}f''(0)\right| + \left|f'(0)\right|^2 \leq 1,$$

which is equivalent to

$$\left\|\mathcal{T}\left(f'(0), \frac{f''(0)}{2}\right)\right\| := \left\|\begin{pmatrix} f'(0) & \frac{f''(0)}{2} \\ 0 & f'(0)\end{pmatrix}\right\| \leq 1.$$

Thus we have proved the following theorem.

Theorem 2.29. *Suppose $f : \mathbb{D} \to \mathbb{D}$ is an analytic function with $f(0) = 0$. Then*

$$\left\|\mathcal{T}\left(f'(0), \frac{f''(0)}{2}\right)\right\| \leq 1.$$

We answer the question of the converse in the theorem below.

Theorem 2.30. *If $\alpha, \beta \in \mathbb{C}$ are such that $\|\mathcal{T}(\beta, \alpha)\| \leq 1$, then there exists an analytic map $f : \mathbb{D} \to \mathbb{D}$ such that $f(0) = 0$, $f'(0) = \beta$ and $f''(0)/2 = \alpha$.*

Proof. Let $\alpha, \beta \in \mathbb{C}$ be such that $\|\mathcal{T}(\beta, \alpha)\| \leq 1$ i.e. $|\alpha| + |\beta|^2 \leq 1$. As $\mathcal{D}_{\mathbb{D}}^0$ is a convex and balanced set so without loss of generality we assume $\alpha > 0$ and $|\alpha| + |\beta|^2 = 1$. Define

$$g(z) := \begin{cases} \beta & if\ z = 0 \\ \frac{f(z)}{z} & otherwise. \end{cases}$$

Let ϕ_β denote the automorphism of $\mathbb{D}$ mapping β to 0. From chain rule, we get

$$\left(\phi_\beta \circ g\right)'(0) = \frac{\alpha}{1-|\beta|^2} = 1.$$

Hence $(\phi_\beta \circ g)(z) = e^{i\theta} z$ for some $\theta \in [0, 2\pi)$. Therefore

$$g(z) = \frac{e^{i\theta} z + \beta}{1 + \overline{\beta} e^{i\theta} z}.$$

and thus

$$f(z) = z \cdot \frac{e^{i\theta} z + \beta}{1 + \overline{\beta} e^{i\theta} z}.$$

□

Thus we have found necessary and sufficient condition for the CF problem 2.28. A second approach to this problem will be given in Chapter 4.

2.4.2 CF interpolation problem in two variables

The complete solution to the CF problem remains a mystery, although, several different partial answers are known. On the other hand, Eschmeier, Patton and Putinar [EPP00] find a necessary and sufficient condition for the CF problem for the bi-disc $\mathbb{D}^2$. However, these conditions are somewhat intractable.

Theorem 2.31. *Let d be a positive integer and let $P(z)$ be a polynomial of degree less than or equal to d in two complex variables. There exists an analytic function $F : \mathbb{D}^2 \to \mathbb{D}$ such that $P \equiv F \mod (z^{d+1})$ if and only if there are Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$ and a pair of vector valued polynomial functions of degree less than or equal to d, $A_k : \mathbb{D}^2 \to \mathbb{H}_k$, $k = 1, 2$, such that:*

$$1 - P(z)P(z) \equiv (1-|z_1|^2)\|A_1(z)\|_1^2 + (1-|z_2|^2)\|A_2(z)\|_2^2 \mod (z^{d+1}, \overline{z}^{d+1}).$$

Analogous to the case of one variable, the CF problem in the case of two variables is given below for polynomials of degree at most two with constant term zero. This is typical of all other cases.

Problem 2.32. Fix $p \in \mathbb{C}[Z_1, Z_2]$ to be the polynomial

$$p(z_1, z_2) = a_{1,0} z_1 + a_{0,1} z_2 + a_{2,0} z_1^2 + a_{1,1} z_1 z_2 + a_{0,2} z_2^2.$$

Find necessary and sufficient conditions for the existence of a holomorphic function q on $\mathbb{D}^2$ with $q^{(k)}(0) = 0$, for multi indices k of length at most 2, such that $\|p + q\|_{\mathbb{D}^2,\infty} \le 1$.

Let T_{x_1}, T_{x_2} be operators of type VI for x_1, x_2 in $\mathbb{C}$, $|x_1|, |x_2| \leq 1$. Let $f \in \mathrm{H}_0^\infty(\mathbb{D}^2, \mathbb{D})$ be any holomorphic function mapping $\mathbb{D}^2$ to $\mathbb{D}$ with $f(0) = 0$. The von-Neumann inequality in Theorem 2.10 implies that $\|f(T_{x_1}, T_{x_2})\| \leq 1$, which in turn is equivalent to

$$\|Df(0)\cdot x\|^2 + \left|\mathrm{tr}\left(\frac{1}{2}D^2 f(0)\cdot x x^{\mathrm{t}}\right)\right| \leq 1,$$

where x is the column vector $\binom{x_1}{x_2}$. Thus we have the following theorem.

Theorem 2.33. *If p is any complex valued polynomial in two variables of degree at most 2 with $p(0) = 0$, then*

$$\sup_{x_1, x_2 \in \mathbb{D}} \left\| \mathcal{T}\left(\frac{\partial p}{\partial z_1}(0)x_1 + \frac{\partial p}{\partial z_2}(0)x_2, \frac{1}{2}\sum_{i,j=1}^{2} \frac{\partial^2 p}{\partial z_i \partial z_j}(0)x_i x_j\right)\right\| \leq 1 \tag{2.23}$$

is a necessary condition for the existence of a holomorphic function $q : \mathbb{D}^2 \to \mathbb{C}$, with $q^{(k)}(0) = 0$, $|k| \leq 2$, such that $\|p + q\|_{\mathbb{D}^2,\infty} \leq 1$.

In the Chapter 3 we will compute the supremum occurring in (2.23). We will also find conditions on the coefficients of the polynomial p, apart from the ones imposed by (2.23), which will ensure the existence of the required function q.

3 Varopoulos Operators of Type II

3.1 Homomorphisms induced by operators of type $\mathsf{V\,II}$ and order two

Let Ω be a bounded domain in $\mathbb{C}^m$ and $\omega = (\omega_1, \ldots, \omega_m) \in \Omega$ be fixed. Let $\mathbb{H}$ be a separable Hilbert space and $X = (X_1, \ldots, X_m)$ be a tuple of commuting contractions, $X_j \in \mathscr{B}(\mathbb{H})$, $j = 1, \ldots, m$. Let T_{X_j} be of type $\mathsf{V\,II}$ and of order 2, $j = 1, \ldots, m$. Let $\boldsymbol{T}_X$ be the n-tuple $(\omega_1 I + T_{X_1}, \ldots, \omega_m I + T_{X_m})$. For $j, k, l = 1, \ldots, m$, we have $T_{X_j} T_{X_k} = T_{X_k} T_{X_j}$, $T_{X_j} T_{X_k} T_{X_l} = 0$ and

$$T_{X_j} T_{X_k} = \begin{pmatrix} 0 & 0 & X_j X_k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let A_X denote the block matrix $\big(\!\big(X_j X_k\big)\!\big)_{m \times m}$ of operators. Consequently, for any polynomial p in m variables, we see that

$$p(\boldsymbol{T}_X) = \begin{pmatrix} p(\omega) I & Dp(\omega) \cdot X & \frac{1}{2} D^2 p(\omega) \cdot A_X \\ 0 & p(\omega) I & Dp(\omega) \cdot X \\ 0 & 0 & p(\omega) I \end{pmatrix}.$$

Therefore, extending this definition to functions in $\mathrm{H}^\infty(\Omega)$, we obtain the algebra homomorphism $\mu_X^{(\omega)} : \mathrm{H}^\infty(\Omega) \to \mathscr{B}(\mathbb{H} \oplus \mathbb{H} \oplus \mathbb{H})$, which for any polynomial p is given by the formula $\mu_X^{(\omega)}(p) = p(\boldsymbol{T}_X)$ and is defined for f in $\mathrm{H}^\infty(\Omega)$ by the same formula. Suppose Ω is the polydisc $\mathbb{D}^m$. Then, for $m = 1, 2$, we know that $\mu_X^{(0)} := \mu_X$ is a contractive homomorphism. What about $m > 2$?

Consider the operators A_1, A_2 and A_3 as defined in the Section 2.3. Consider T_{A_1}, T_{A_2} and T_{A_3}, the operators of type $\mathsf{V\,II}$ and of order 2. Consider the Varopoulos-Kaijser polynomial $p_{_V}$. From the computation in [Var74], we get

$$\big\| p_{_V}(T_{A_1}, T_{A_2}, T_{A_3}) \big\| = \Big\| \sum_{j,k=1}^{3} a_{jk} A_k A_j \Big\| = 3\sqrt{3},$$

where $(\!(a_{jk})\!) = A_V$. Therefore

$$\left\| p_V(T_{A_1}, T_{A_2}, T_{A_3}) \right\| > \| p_V \|_{\mathbb{D}^3,\infty} = 5.$$

Hence μ_X corresponding to the tuple $X = (A_1, A_2, A_3)$ of commuting contractions A_1, A_2 and A_3 is not contractive.
We need a version of "the zero lemma" one final time which is adapted to apply directly to the functional calculus for operators of the type V II. This variant is also proved exactly the same way as before.

Lemma 3.1. *The homomorphism $\mu_X^{(\omega)}$ is contractive if and only if $\|\mu_X^{(\omega)}(f)\| \le 1$ for all f in $\mathrm{H}^\infty_\omega(\Omega, \mathbb{D})$.*

If Ω is the unit disc $\mathbb{D}$ and $\omega = 0$, then we have the following theorem.

Theorem 3.2. *Suppose $\mathbb{H}$ is a separable Hilbert space and $X \in \mathscr{B}(\mathbb{H})$ with $\|X\| \le 1$. Then for homomorphism μ_X, we get* $\sup\left\{\|\mu_X(p)\| : p \in \mathscr{P}_1^0(\mathbb{D}, \mathbb{D})\right\} = \sup\left\{\|\mu_{\boldsymbol{X}}(f)\| : f \in \mathrm{H}_0^\infty(\mathbb{D}, \mathbb{D})\right\}$.

Proof. We have $\sup\left\{\|\mu_X(p)\| : p \in \mathscr{P}_1^0(\mathbb{D}, \mathbb{D})\right\} = \|X\|$ by definition. Fix $f \in \mathrm{H}_0^\infty(\mathbb{D}, \mathbb{D})$, and assume that f is represented in the unit disc $\mathbb{D}$ by the convergent power series $\sum_{j=1}^\infty a_n z^n$. Then

$$\|\mu_X(f)\| = \left\|\mathfrak{T}\left(a_1 X, a_2 X^2\right)\right\| \le \|X\| \, \|\mathfrak{T}(a_1 I, a_2 X)\|.$$

Also $\|f\|_{\mathbb{D},\infty} \ge \|\mathfrak{T}(a_1, a_2)\|$, therefore

$$\frac{\|\mu_X(f)\|}{\|f\|_{\mathbb{D},\infty}} \le \frac{\|X\| \, \|\mathfrak{T}(a_1 I, a_2 X)\|}{\|\mathfrak{T}(a_1, a_2)\|} \le \|X\|.$$

Hence

$$\sup\left\{\|\mu_X(p)\| : p \in \mathscr{P}_1^0(\mathbb{D}, \mathbb{D})\right\} = \sup\left\{\|\mu_X(f)\| : f \in \mathrm{H}_0^\infty(\mathbb{D}, \mathbb{D})\right\} = \|X\|.$$

□

For two commuting contractions $X_1, X_2 \in \mathscr{B}(\mathbb{H})$, let $X = (X_1, X_2)$ and T_{X_1}, T_{X_2} be the operators of type V II and order 2. Setting $\Omega = \mathbb{D}^2$, and $\omega = 0$, we see that the homomorphism μ_X is contractive via Ando's theorem. Hence for $f \in \mathrm{H}_0^\infty(\mathbb{D}^2, \mathbb{D})$, we have

$$\sup_X \left\| \mathfrak{T}\left(Df(0) \cdot X, \frac{1}{2} D^2 f(0) \cdot A_X \right) \right\| \le 1,$$

where the supremum is taken over all pairs of commuting contractions $X = (X_1, X_2)$. Thus we have proved the following theorem.

Theorem 3.3. *Let $f \in H_0^\infty(\mathbb{D}^2, \mathbb{C})$. Then*

$$\sup \left\| \mathfrak{T}\left(\frac{\partial f}{\partial z_1}(0) X_1 + \frac{\partial f}{\partial z_2}(0) X_2, \ \frac{1}{2} \sum_{i,j=1}^{2} \frac{\partial^2 f}{\partial z_i \partial z_j}(0) X_i X_j \right) \right\| \leq 1,$$

where the supremum is taken over all pairs of commuting contractions $X_1, X_2 \in \mathscr{B}(\mathbb{H})$, is a necessary condition for f to map $\mathbb{D}^2$ to $\mathbb{D}$.

In what follows, we show that the supremum in (2.23) is the same as the one appearing in Theorem 3.3. We then proceed to compute it explicitly. Let $(B)_1$ denote the open unit ball of the Banach space B. Let Ω be a domain in $\mathbb{C}^m$ and $k \in \mathbb{N}$. We shall denote the set of all M_k−valued polynomials in m variables by $\mathscr{P}(\mathbb{C}^m, M_k)$. The symbol $\mathscr{P}_n(\mathbb{C}^m, M_k)$ will denote the set of all polynomials in $\mathscr{P}(\mathbb{C}^m, M_k)$ which are of degree at most n. For any $\omega \in \Omega$, we shall denote $\mathscr{P}_n^{(\omega)}(\mathbb{C}^m, M_k) := \{p \in \mathscr{P}_n(\mathbb{C}^m, M_k) : p(\omega) = 0\}$ and $\mathscr{P}_n^{(\omega)}(\Omega, (M_k)_1) := \left\{ p \in \mathscr{P}_n^{(\omega)}(\mathbb{C}^m, M_k) : \|p\|_{\Omega,\infty}^{\mathrm{op}} \leq 1 \right\}$, where $\|p\|_{\Omega,\infty}^{\mathrm{op}} = \sup\{\|p(z)\|_{op} : z \in \Omega\}$.

Our attempt here to find a solution to the extremal problem stated in the Theorem 3.3 leads naturally to an independent verification of the von-Neumann inequality for pairs of commuting contractions of type V II.

3.2 The von-Neumann Inequality in One variable

In what follows the following lemma is needed.

Lemma 3.4. *If $F(z) = A_1 + A_2 z + A_3 z^2 + \cdots$ is an analytic map on $\mathbb{D}$ taking values in $(M_k)_1$, then $\mathfrak{T}(A_1, A_2)$ has norm at most 1.*

Proof. Suppose $\phi_{-A_1} : (M_k)_1 \to (M_k)_1$ is an analytic map defined by

$$\phi_{-A_1}(C) = \left(I - A_1 A_1^*\right)^{-\frac{1}{2}} (C - A_1)\left(I - A_1^* C\right)^{-1}\left(I - A_1^* A_1\right)^{\frac{1}{2}}.$$

Then $\phi_{-A_1} \circ F$ maps $\mathbb{D}$ to $(M_k)_1$ with $(\phi_{-A_1} \circ F)(0) = 0$.

$$\left(\phi_{-A_1} \circ F\right)'(0) = \phi'_{-A_1}\big(F(0)\big) F'(0) = \left(I - A_1 A_1^*\right)^{-\frac{1}{2}} A_2 \left(I - A_1^* A_1\right)^{-\frac{1}{2}}$$

Schwarz's lemma implies that $(\phi_{-A_1} \circ F)'(0)$ is a contractive linear map from $\mathbb{C}$ to M_k and therefore

$$\left\| \left(I - A_1 A_1^*\right)^{-\frac{1}{2}} A_2 \left(I - A_1^* A_1\right)^{-\frac{1}{2}} \right\| \leq 1.$$

Now due to Parrott's theorem, we conclude that $\|\mathfrak{T}(A_1, A_2)\| \leq 1$. □

The theorem below proves the von-Neumann inequality (involving matrix valued polynomials) for operators of type VII and order 2.

The homomorphism $\mu_X^{(\omega)}$ naturally extends to the algebra $\mathrm{H}^\infty(\Omega)\otimes M_k$ by tensoring with the identity map I_k on the $k\times k$ matrices M_k. Thus

$$\mu_X^{(\omega)}\otimes I_k : \mathrm{H}^\infty(\Omega)\otimes M_k \to \mathscr{B}(\mathbb{H}\otimes\mathbb{C}^3)\otimes M_k$$

is given by the formula $\mu_X^{(\omega)}\otimes I_k(F) := \big(\!\big(\mu_X^{(\omega)}(F_{ij})\big)\!\big)$, where $F = \big(\!\big(F_{ij}\big)\!\big) \in \mathrm{H}^\infty(\Omega)\otimes M_k$. In particular, for any $F(z) = A_0 + A_1 z + A_2 z^2 + \cdots$, it follows that

$$\begin{aligned}\mu_X^{(\omega)}\otimes I_k(F) &= A_0\otimes I_k + A_1\otimes T_X + A_2\otimes T_X^2+\cdots\\ &= \begin{pmatrix} A_0 & A_1\otimes X & A_2\otimes X^2\\ 0 & A_0 & A_1\otimes X\\ 0 & 0 & A_0\end{pmatrix}\end{aligned}\tag{3.1}$$

Theorem 3.5. *Let X be a contraction on some Hilbert space $\mathbb{H}$ and T_X be the operator of type VII and order 2. If $P \in \mathscr{P}_n(\mathbb{D},(M_k)_1)$ then, $\|P(T_X)\|\le 1$.*

Proof. The zero lemma 3.1 is easy to prove for the homomorphism $\mu_X^{(\omega)}\otimes I_k$. The proof now involves finding an automorphism of the unit ball (with respect to the operator norm) in the $k\times k$ matrices taking $P(\omega)$ to 0. Since this group of automorphisms is known to be transitive, the proof of the zero lemma even for matrix valued polynomials is same in spirit to the ones given before. We therefore assume, without loss generality, that $P(0)=0$. Thus for any polynomial $P(z) = A_1 z + A_2 z^2 + \cdots + A_n z^n$, we have

$$P(T_X) := \mu_X\otimes I_k(P) = \begin{pmatrix} 0 & A_1\otimes X & A_2\otimes X^2\\ 0 & 0 & A_1\otimes X\\ 0 & 0 & 0\end{pmatrix}$$

and hence

$$\begin{aligned}\|P(T_X)\| &= \left\|\begin{pmatrix} A_1\otimes X & A_2\otimes X^2\\ 0 & A_1\otimes X\end{pmatrix}\right\|\\ &= \left\|\begin{pmatrix} I\otimes X & 0\\ 0 & I\end{pmatrix}\begin{pmatrix} A_1\otimes I & A_2\otimes I\\ 0 & A_1\otimes I\end{pmatrix}\begin{pmatrix} I & 0\\ 0 & I\otimes X\end{pmatrix}\right\|.\end{aligned}$$

Since $\|X\|\le 1$, it follows that $\|P(T_X)\| \le \|\mathfrak{T}(A_1,A_2)\|$. Now using the Lemma 3.4, we get $\|P(T_X)\|\le 1$. $\square$

3.3 Ando's Theorem for the Operators of type V II and order 2

The zero lemma 3.1, modified as in the previous section, applies to the case of matrix valued polynomials in any number of variables. In this form, it is stated in the papers [MNS90, Pau92]. Consequently it is enough, without loss of generality, that the homomorphisms we consider below are defined only on polynomials with $P(0) = 0$. We now recall the commutant lifting theorem (cf.[DMP68, Theorem 4]), which we use in the proof of the lemma below.

Theorem 3.6. *Let T be a contraction on a Hilbert space $\mathbb{H}$, U be its minimal co-isometric dilation acting on some Hilbert space $\mathbb{K}$, and R be an operator on $\mathbb{H}$ commuting with T. Then there is an operator S on $\mathbb{K}$ commuting with U such that*

$$S\mathbb{H} \subset \mathbb{H},\ \|S\| = \|R\| \ \textit{and}\ R^m T^n = P_{\mathbb{H}} S^m U^n|_{\mathbb{H}}\ \forall m,n \geq 0.$$

Let X_1 and X_2 be two commuting contractions on a Hilbert space $\mathbb{H}$ and T_{X_1}, T_{X_2} be the operators of type V II and order 2 respectively. Let

$$P(z_1,z_2) = \sum_{k=1}^{n} \sum_{p+q=k} A_{pq} z_1^p z_2^q$$

be in $\mathscr{P}_n^{(0)}(\mathbb{C}^2, M_l)$. Evaluating the polynomial P on the commuting pair of contractions T_{X_1}, T_{X_2}, we get

$$P(T_{X_1},T_{X_2}) = \begin{pmatrix} 0 & \sum\limits_{p+q=1} A_{pq}\otimes X_1^p X_2^q & \sum\limits_{p+q=2} A_{pq}\otimes X_1^p X_2^q \\ 0 & 0 & \sum\limits_{p+q=1} A_{pq}\otimes X_1^p X_2^q \\ 0 & 0 & 0 \end{pmatrix}$$

and hence

$$\|P(T_{X_1},T_{X_2})\| = \left\| \begin{pmatrix} \sum\limits_{p+q=1} A_{pq}\otimes X_1^p X_2^q & \sum\limits_{p+q=2} A_{pq}\otimes X_1^p X_2^q \\ 0 & \sum\limits_{p+q=1} A_{pq}\otimes X_1^p X_2^q \end{pmatrix} \right\|.$$

Lemma 3.7. *For any polynomial P in $\mathscr{P}_n^{(0)}(\mathbb{C}^2, M_l)$, $\|P(T_{X_1},T_{X_2})\| \leq 1$ for all commuting contractions X_1, X_2 if and only if $\|P(T_{X_1},T_{X_2})\| \leq 1$ for all commuting pairs X_1, X_2 with X_1 is co-isometry and X_2 contractive.*

Proof. Let X_1 and X_2 be any two commuting contractions. Let $U : \mathbb{K} \to \mathbb{K}$ be the minimal co-isometric dilation of X_1. From commutant lifting theorem there exists an operator $S : \mathbb{K} \to \mathbb{K}$ such that

$$\|S\| = \|X_2\|,\ SU = US \text{ and } X_2^m {X_1}^n = P_{\mathbb{H}} S^m U^n|_{\mathbb{H}} \text{ for all } m, n \in \mathbb{N}_0.$$

Let T_U and T_S be the operators of type VII and order 2. Setting $\widetilde{\mathbb{H}} = \mathbb{C}^l \otimes \mathbb{H} \oplus \mathbb{C}^l \otimes \mathbb{H}$, we have

$$P_{\widetilde{\mathbb{H}}} \mathfrak{T}\big(A_{10} \otimes U + A_{01} \otimes S, A_{20} \otimes U^2 + A_{11} \otimes US + A_{02} \otimes S^2\big)\big|_{\widetilde{\mathbb{H}}}$$

$$= \mathfrak{T}\left(\sum_{p+q=1} A_{pq} \otimes X_1^p X_2^q, \sum_{p+q=2} A_{pq} \otimes X_1^p X_2^q\right).$$

Thus for any polynomial P of the form

$$P(z_1, z_2) = \sum_{k=1}^{n} \sum_{p+q=k} A_{pq} z_1^p z_2^q$$

mapping $\mathbb{D}^2$ into $(M_l)_1$,, we have $\|P(T_U, T_S)\| \geq \|P(T_{X_1}, T_{X_2})\|$ completing the proof of the lemma. □

Theorem 3.8. *Let T_{X_1} and T_{X_2} be commuting contractions of type VII and order 2. We have $\|P(T_{X_1}, T_{X_2})\| \leq \|P\|^{\mathrm{op}}_{\mathbb{D}^2,\infty}$, for any matrix valued polynomial P in two variables.*

Proof. Let P be a polynomial in two variables of the form

$$P(z_1, z_2) = \sum_{k=1}^{n} \sum_{p+q=k} A_{pq} z_1^p z_2^q,\ A_{pq} \in M_l,$$

with $\|P\|^{\mathrm{op}}_{\mathbb{D}^2,\infty} \leq 1$. For $\lambda \in \mathbb{D}$,

$$p_\lambda(z_1) := P(z_1, \lambda z_1) = (A_{10} + A_{01}\lambda) z_1 + (A_{20} + A_{11}\lambda + A_{02}\lambda^2) {z_1}^2 + \ldots$$

maps $\mathbb{D}$ to $(M_l)_1$. Therefore for each $\lambda \in \mathbb{D}$,

$$\left\| \begin{pmatrix} A_{10} + A_{01}\lambda & A_{20} + A_{11}\lambda + A_{02}\lambda^2 \\ 0 & A_{10} + A_{01}\lambda \end{pmatrix} \right\| \leq 1,$$

which is equivalent to

$$\left\| \mathfrak{T}(A_{10}, A_{20}) + \mathfrak{T}(A_{01}, A_{11})\,\lambda + \mathfrak{T}(0, A_{02})\,\lambda^2 \right\| \leq 1,$$

for all $\lambda \in \mathbb{D}$. Define $f : \mathbb{D} \to (M_{2l})_1$ by

$$f(\lambda) = \mathfrak{T}(A_{10}, A_{20}) + \mathfrak{T}(A_{01}, A_{11})\lambda + \mathfrak{T}(0, A_{02})\lambda^2.$$

Let X_1 be a co-isometry operator and X_2 be an arbitrary contraction such that $X_1X_2 = X_2X_1$. If $Y := X_2X_1^*$, then

$$f(Y) = \mathfrak{T}\left(A_{10} \otimes I + A_{01} \otimes Y,\ A_{20} \otimes I + A_{11} \otimes Y + A_{02} \otimes Y^2\right).$$

An easy computation gives

$$\begin{aligned} &\begin{pmatrix} A_{10} \otimes X_1 + A_{01} \otimes X_2 & A_{20} \otimes {X_1}^2 + A_{11} \otimes X_2X_1 + A_{02} \otimes {X_2}^2 \\ 0 & A_{10} \otimes X_1 + A_{01} \otimes X_2 \end{pmatrix} \\ &= \begin{pmatrix} I \otimes X_1 & 0 \\ 0 & I \end{pmatrix} f(Y) \begin{pmatrix} I & 0 \\ 0 & I \otimes X_1 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\left\| \mathfrak{T}\left(\sum_{p+q=1} A_{pq} \otimes X_1^p X_2^q,\ \sum_{p+q=2} A_{pq} \otimes X_1^p X_2^q \right) \right\| \le \| f(X_2{X_1}^*) \|$$

and since $X_2X_1^*$ is a contraction, therefore by the von-Neumann inequality, we have

$$\left\| \mathfrak{T}\left(\sum_{p+q=1} A_{pq} \otimes X_1^p X_2^q,\ \sum_{p+q=2} A_{pq} \otimes X_1^p X_2^q \right) \right\| \le 1.$$

This completes the proof of the theorem. □

3.4 Solution to the Extremal Problem

In this section we shall calculate the supremum occurring in Theorem 2.33 and Theorem 3.3.

Let B be the bilateral shift on $\ell^2(\mathbb{Z})$ and $C^*(B)$ be the commutative unital C^*- algebra generated by B. Hence $C^*(B)$ is isometrically isomorphic to the C^*- algebra of continuous functions $C(\sigma(B))$, where $\sigma(B) = \mathbb{T}$ is the unit circle in $\mathbb{C}$. This isometric isomorphism, which we denote by τ, is defined by the rule $\tau(f) = f(B^*)$. Consequently, for any $k \in \mathbb{N}$, the map

$$\tau \otimes I_k : C(\mathbb{T}) \otimes M_k \to \mathscr{B}(\ell^2(\mathbb{Z})) \otimes M_k$$

is also a $*-$ isometric monomorphism. In particular, for any $P \in \mathscr{P}(\mathbb{C}, M_k)$, we have

$$\|P\|_{\mathbb{D},\infty}^{\mathrm{op}} = \|P(B^*)\|. \tag{3.2}$$

In the proof of the Theorem 3.8, we have seen that in solving the extremal problem

$$\sup\left\|\mathcal{T}\left(\frac{\partial f}{\partial z_1}(0)X_1+\frac{\partial f}{\partial z_2}(0)X_2,\ \frac{1}{2}\sum_{i,j=1}^{2}\frac{\partial^2 f}{\partial z_i\partial z_j}(0)X_iX_j\right)\right\|$$

over all commuting contractions $X_1, X_2 \in \mathscr{B}(\mathbb{H})$, we may assume without loss of generality that $X_1 = I$. Therefore the supremum in Theorem 3.3 is equal to the

$$\sup\left\|\mathcal{T}\left(\frac{\partial f}{\partial z_1}(0)I+\frac{\partial f}{\partial z_2}(0)X,\ \frac{1}{2}\left(\frac{\partial^2 f}{\partial z_1^2}(0)I+2\frac{\partial^2 f}{\partial z_1\partial z_2}(0)X+\frac{\partial^2 f}{\partial z_2^2}(0)X^2\right)\right)\right\|, \tag{3.3}$$

where the supremum is taken over all contractions X.

Let P be the polynomial (taking values in 2×2 matrices M_2)

$$P(\lambda)=\mathcal{T}\left(\frac{\partial f}{\partial z_1}(0),\frac{1}{2}\frac{\partial^2 f}{\partial z_1^2}(0)\right)+\mathcal{T}\left(\frac{\partial f}{\partial z_2}(0),\frac{\partial^2 f}{\partial z_1\partial z_2}(0)\right)\lambda+\mathcal{T}\left(0,\frac{1}{2}\frac{\partial^2 f}{\partial z_2^2}(0)\right)\lambda^2$$

We have

$$\sup_{x_1,x_2\in\mathbb{D}}\left\|\mathcal{T}\left(\frac{\partial f}{\partial z_1}(0)x_1+\frac{\partial f}{\partial z_2}(0)x_2,\frac{1}{2}\sum_{i,j=1}^{2}\frac{\partial^2 f}{\partial z_i\partial z_j}(0)x_ix_j\right)\right\|=\|P\|^{\mathrm{op}}_{\mathbb{D},\infty}$$

and for any contraction X, an application of the von-Neumann inequality gives $\|P(X)\| \le \|P\|^{\mathrm{op}}_{\mathbb{D},\infty}$. From (3.2) we have, $\|P\|^{\mathrm{op}}_{\mathbb{D},\infty} = \|P(B^*)\|$. Therefore, $\sup\|P(X)\| = \|P(B^*)\|$ and hence supremum in (3.3), Theorem 2.33 and Theorem 3.3 are equal to

$$\left\|\mathcal{T}\left(\frac{\partial f}{\partial z_1}(0)I+\frac{\partial f}{\partial z_2}(0)B^*,\ \frac{1}{2}\frac{\partial^2 f}{\partial z_1^2}(0)I+\frac{\partial^2 f}{\partial z_1\partial z_2}(0)B^*+\frac{1}{2}\frac{\partial^2 f}{\partial z_2^2}(0)B^{*2}\right)\right\|.$$

Thus Theorem 2.33 and Theorem 3.3 are equivalent to the following theorem.

Theorem 3.9. *For any $f\in \mathrm{H}^\infty_0(\mathbb{D}^2,\mathbb{D})$,*

$$\left\|\mathcal{T}\left(\frac{\partial f}{\partial z_1}(0)I+\frac{\partial f}{\partial z_2}(0)B^*,\ \frac{1}{2}\frac{\partial^2 f}{\partial z_1^2}(0)I+\frac{\partial^2 f}{\partial z_1\partial z_2}(0)B^*+\frac{1}{2}\frac{\partial^2 f}{\partial z_2^2}(0)B^{*2}\right)\right\|\le 1.$$

The following corollary is essentially a restatement of Theorem 3.9. However, it is worded to make the necessary condition for the CF problem inherent in this theorem, evident.

Corollary 3.10. *If p is any complex valued polynomial in two variables of degree at most 2 with $p(0)=0$, then*

$$\left\|\mathcal{T}\left(\frac{\partial p}{\partial z_1}(0)I+\frac{\partial p}{\partial z_2}(0)B^*,\frac{1}{2}\frac{\partial^2 p}{\partial z_1^2}(0)I+\frac{\partial^2 p}{\partial z_1\partial z_2}(0)B^*+\frac{1}{2}\frac{\partial^2 p}{\partial z_2^2}(0)B^{*2}\right)\right\|\le 1$$

is a necessary condition for the existence of a holomorphic function $q:\mathbb{D}^2\to\mathbb{C}$, with $q^{(k)}(0)=0$, $|k|\le 2$, such that $\|p+q\|_{\mathbb{D}^2,\infty}\le 1$.

In the Chapter 4, we will give another proof of Theorem 3.9 and investigate the question of the converse.

4 The Korányi-Pukánszky Theorem and CF Problems

4.1 The Korányi-Pukánszky Theorem

We recall the following theorem of Korányi and Pukánszky proved in [KP63, Corollary, Page 452]. This gives a necessary and sufficient condition for the range of a holomorphic function defined on the polydisc $\mathbb{D}^n$ to be in the right half plane H_+.

Theorem 4.1 (Korányi-Pukánszky Theorem)**.** *If the power series $\sum_{\alpha\in\mathbb{N}_0^n} a_\alpha z^\alpha$ represents a holomorphic function f on the polydisc $\mathbb{D}^n$, then $\Re(f(z)) \geq 0$ for all $z \in \mathbb{D}^n$ if and only if the map $\phi : \mathbb{Z}^n \to \mathbb{C}$ defined by*

$$\phi(\alpha) = \begin{cases} 2\Re a_\alpha & \text{if } \alpha = 0 \\ a_\alpha & \text{if } \alpha > 0 \\ a_{-\alpha} & \text{if } \alpha < 0 \\ 0 & \text{otherwise} \end{cases}$$

is positive, that is, the $k \times k$ matrix $\left(\!\left(\phi(m_i - m_j)\right)\!\right)$ is non-negative definite for every choice of $m_1, \ldots, m_k \in \mathbb{Z}^n$.

We will call the function ϕ, the Korányi-Pukánszky function corresponding to the coefficients $(a_\alpha)_{\alpha\in\mathbb{N}_0^n}$.

Let us revisit the (CF) problem of realizing a polynomial $p \in \mathbb{C}[Z_1, \ldots, Z_n]$ of degree d as the first d terms of the power series expansion of an analytic function $f \in \mathrm{H}^\infty(\mathbb{D}^n)$ with $\|f\|_{\mathbb{D}^n,\infty} \leq 1$.

4.2 The planar Case

Although, we state the problem below for polynomials of degree 2, our methods apply to the general case.

Problem 4.2. Fix p to be the polynomial $p(z) = az + bz^2$. Find a necessary and sufficient condition for the existence of a holomorphic function g defined on the unit disc $\mathbb{D}$ with $g^{(k)}(0) = 0$, $k = 0, 1, 2$, such that $\|p + g\|_{\mathbb{D},\infty} \le 1$.

We note that the condition on the range of a holomorphic function given in the theorem of Korányi and Pukánszky can be easily converted into a condition where the range is required to be in the unit disc $\mathbb{D}$. For this consider the Cayley map $\chi : \mathbb{D} \to H_+$ into the right half plane defined by

$$\chi(z) = \frac{1+z}{1-z},$$

which is a bi-holomorphism. Let $f \in \mathrm{H}^\infty(\mathbb{D})$ be given by the power series $f(z) = \sum_{n=1}^\infty a_n z^n$. Assume that f maps $\mathbb{D}$ to $\mathbb{D}$. This happens if and only if $\chi \circ f$ maps $\mathbb{D}$ to H_+. Also,

$$\chi \circ f(z) = \frac{1+f(z)}{1-f(z)} = 2\left(c_0 + \sum_{n=1}^\infty c_n z^n\right), \tag{4.1}$$

where $c_0 = 1/2$ and the new coefficients c_n are as in the lemma below. In this section, we set $c_0 = 1/2$, wherever it occurs.

Lemma 4.3. *The coefficient c_n in equation (4.1) is given by $a_n + \sum_{j=1}^{n-1} a_j c_{n-j}$ for $n \ge 1$.*

Proof. Consider the expression

$$\chi \circ f(z) = 2\left(c_0 + \sum_{n=1}^\infty c_n z^n\right) = 2\left(\frac{1}{2} + f(z) + f(z)^2 + f(z)^3 + \cdots\right).$$

Rewriting, we get

$$\frac{1}{1-f(z)} = 1 + \sum_{n=1}^\infty c_n z^n.$$

Hence, we have

$$\left(1 + \sum_{n=1}^\infty c_n z^n\right)\left(1 - \sum_{n=1}^\infty a_n z^n\right) = 1.$$

A comparison of the coefficients completes the verification. □

Remark 4.4. *Applying Theorem 4.1 to $\chi \circ f$, we conclude that f maps $\mathbb{D}$ to $\mathbb{D}$ if and only if the Korányi-Pukánszky function ϕ corresponding to coefficients $(c_n)_{n=0}^\infty$ is positive.*

Matrix of ϕ : The matrix $\big(\phi(j-k)\big)_{j,k}$ is given by

$$\begin{array}{c|ccccc} & \cdots & -1 & 0 & 1 & \cdots \\ \hline \vdots & & & & & \\ -1 & & & & & \\ 0 & & & & & \\ 1 & & & & & \\ \vdots & & & & & \end{array}\qquad \begin{array}{c} \\ \begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & 1 & \bar{c}_1 & \bar{c}_2 & \cdots \\ \cdots & c_1 & 1 & \bar{c}_1 & \cdots \\ \cdots & c_2 & c_1 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix}. \end{array} \tag{4.2}$$

Therefore, we can rewrite the Problem 4.2 in the equivalent form:

Problem 4.5. Let p be a polynomial of the form $p(z) = a_1 z + a_2 z^2$. There exists a holomorphic function q, $q^{(k)}(0) = 0$ for $k = 0,1,2$, defined on the unit disc $\mathbb{D}$, such that $\|p+q\|_{\mathbb{D},\infty} \le 1$ if and only if

$$\begin{pmatrix} 1 & \bar{c}_1 & \bar{c}_2 \\ c_1 & 1 & \bar{c}_1 \\ c_2 & c_1 & 1 \end{pmatrix}$$

is non-negative definite and for $j > 2$, there exists $c_j \in \mathbb{C}$ such that the Korányi-Pukánszky function ϕ corresponding to $(c_n)_{n\in\mathbb{N}_0}$ is positive.

4.3 Alternative proof of Theorem 2.30

Suppose f is an analytic function on the unit disc $\mathbb{D}$ with $\|f\|_{\mathbb{D},\infty} \le 1$ and that $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is its power series expansion in the unit disc $\mathbb{D}$. Then $\chi \circ f(z)$ has the power series $2(c_0 + \sum_{n=1}^{\infty} c_n z^n)$ in the unit disc $\mathbb{D}$, where $c_0 = 1/2$ and c_n is of the form prescribed in the Lemma 4.3. In this section also, we set $c_0 = 1/2$, wherever it occurs. Let C_n, A_n and P_n denote

$$(C_n :=) \begin{pmatrix} 1 & \bar{c}_1 & \bar{c}_2 & \cdots & \bar{c}_n \\ c_1 & 1 & \bar{c}_1 & \cdots & \bar{c}_{n-1} \\ c_2 & c_1 & 1 & \cdots & \bar{c}_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & c_{n-2} & \cdots & 1 \end{pmatrix}, (A_n :=) \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix}$$

and

$$(P_n :=)\begin{pmatrix} 1 & -a_1 & -a_2 & \cdots & -a_n \\ 0 & 1 & -a_1 & \cdots & -a_{n-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -a_1 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

respectively for each $n \in \mathbb{N}$.

Lemma 4.6. *If $a_1, a_2 \in \mathbb{C}$, then $|a_1|^2 + |a_2| \le 1$ if and only if the matrix C_2 is non-negative definite.*

Proof. Since $|a_1|^2 + |a_2| \le 1$, it follows that $\|A_2\| \le 1$. It is equivalent to the positivity of the following matrix

$$\begin{pmatrix} I - A_2 A_2^* & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-(|a_1|^2+|a_2|^2) & -a_2\overline{a}_1 & 0 \\ -a_1\overline{a}_2 & 1-|a_1|^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = P_2 C_2^{\mathrm{t}} P_2^*.$$

Since P_2 is an invertible matrix, the positivity of $P_2 C_2^{\mathrm{t}} P_2^*$ is equivalent to C_2^{t} being non-negative definite. Thus, we conclude that C_2 is non-negative definite. □

Lemma 4.7. *For all $n \in \mathbb{N}$, $P_n C_n^{\mathrm{t}} P_n^* = (I - A_n A_n^*) \oplus 1$.*

Proof. We shall prove the result by induction on n. The case $n = 1$ follows from the Lemma 4.6. Assume the result upto $n-1$ for $n > 1$. For each $n \in \mathbb{N}$, let

$$\tilde{P}_n := (-a_n, -a_{n-1}, \ldots, -a_1)^{\mathrm{t}} \text{ and } \tilde{C}_n := (c_n, c_{n-1}, \ldots, c_1)^{\mathrm{t}}.$$

The verification of the identity

$$P_n C_n^{\mathrm{t}} P_n^* = \begin{pmatrix} P_{n-1} & \tilde{P}_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C_{n-1}^{\mathrm{t}} & \tilde{C}_n \\ \tilde{C}_n^* & 1 \end{pmatrix} \begin{pmatrix} P_{n-1}^* & 0 \\ \tilde{P}_n^* & 1 \end{pmatrix}$$

is easy. Hence

$$P_n C_n^{\mathrm{t}} P_n^* = \begin{pmatrix} P_{n-1} C_{n-1}^{\mathrm{t}} P_{n-1}^* + \tilde{P}_n \tilde{C}_n^* P_{n-1}^* + \tilde{P}_n^* \left(P_{n-1}\tilde{C}_n + \tilde{P}_n\right) & P_{n-1}\tilde{C}_n + \tilde{P}_n \\ \left(P_{n-1}\tilde{C}_n + \tilde{P}_n\right)^* & 1 \end{pmatrix}.$$

From the Lemma 4.3, we have $P_{n-1}\tilde{C}_n + \tilde{P}_n = 0$ and therefore we conclude that

$$P_n C_n P_n^* = \begin{pmatrix} P_{n-1} C_{n-1}^{\mathrm{t}} P_{n-1}^* + \tilde{P}_n \tilde{C}_n^* P_{n-1}^* & 0 \\ 0 & 1 \end{pmatrix}.$$

Now

$$\tilde{P}_n\tilde{C}_n^*P_{n-1}^* = \begin{pmatrix} -a_n \\ \vdots \\ -a_1 \end{pmatrix}\begin{pmatrix} \overline{c}_n - \sum_{i=1}^{n-1} a_i c_{n-i} & \overline{c}_{n-1} - \sum_{i=1}^{n-2} a_i c_{n-i} & \cdots & \overline{c}_1 \end{pmatrix}.$$

From the Lemma 4.3, we get

$$\tilde{P}_n\tilde{C}_n^*P_{n-1}^* = \begin{pmatrix} -a_n \\ \vdots \\ -a_1 \end{pmatrix}\begin{pmatrix} \overline{a}_n & \cdots & \overline{a}_1 \end{pmatrix} = \left(-a_{n-i}\overline{a}_{n-j}\right)_{i,j=0}^{n-1}.$$

Also,

$$I - A_kA_k^* = \begin{pmatrix} 1-\sum_{j=1}^{k}|a_j|^2 & -\sum_{j=2}^{k} a_j\overline{a}_{j-1} & \cdots & -a_k\overline{a}_1 \\ -\sum_{j=2}^{k}\overline{a}_j a_{j-1} & 1-\sum_{j=1}^{k-1}|a_j|^2 & \cdots & -a_{k-1}\overline{a}_1 \\ \vdots & \vdots & \ddots & \vdots \\ -a_1\overline{a}_k & -a_1\overline{a}_{k-1} & \cdots & 1-|a_1|^2 \end{pmatrix}$$

and therefore $I - A_nA_n^* = \left((I-A_{n-1}A_{n-1}^*)\oplus 1\right) + \left(-a_{n-j}\overline{a}_{n-l}\right)_{1\le j,l\le k-1}$. Thus $I - A_nA_n^* = P_{n-1}C_{n-1}^{\mathrm{t}}P_{n-1}^* + \tilde{P}_n\tilde{C}_nP_{n-1}^*$, which completes the proof. □

An immediate corollary is the following proposition.

Proposition 4.8. *The matrix C_n is non-negative definite if and only if A_n satisfies $\|A_n\| \le 1$.*

In the theorem below we provide an alternative proof of the Theorem 2.30. The technique involved here is from the Section 3 of the paper of Parrott [Par78].

Theorem 4.9. *Suppose a_1 and a_2 are two complex numbers. Then there exists $f \in \mathrm{H}^\infty(\mathbb{D})$, with $\|f\|_{\mathbb{D},\infty} \le 1$, such that $f(0) = 0,\ f'(0) = a_1,\ f''(0) = 2a_2$ if and only if $|a_1|^2 + |a_2| \le 1$.*

Proof. Assume f and $\chi\circ f$ are as in (4.1). Then by Korányi-Pukánszky theorem 4.1, every principal submatrix of finite size of the matrix in (4.2) is non-negative definite. In particular, the matrix C_2 is non-negative definite. From the Lemma 4.6 we conclude that $|a_1|^2 + |a_2| \le 1$. Conversely, assume $a_1, a_2 \in \mathbb{C}$ are such that $|a_1|^2 + |a_2| \le 1$. Then,

$$A_2 = \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix}$$

satisfies $\|A_2\| \le 1$. Using Parrott's theorem, there exists $a_3 \in \mathbb{C}$ such that

$$A_3 = \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_1 & a_2 \\ 0 & 0 & a_1 \end{pmatrix}$$

has operator norm less than or equal to 1. Using the Lemma 4.3, we see that the matrix C_3 is non-negative definite. Repeatedly using the Parrott's theorem and the Lemma 4.3, one may ensure the existence of non-negative definite matrices C_n, for all $n > 3$.

Hence the Korányi-Pukánszky function corresponding to $(c_n)_{\mathbb{N}_0}$ is positive. Thus the function $g(z) = \sum_n c_n z^n$ maps $\mathbb{D}$ to $\mathbb{H}_+$ by Korányi-Pukánszky theorem 4.1. Hence from the Lemma 4.3, the function $f = \chi^{-1} \circ g$ satisfies all the required conditions. □

4.4 The case of two Variables

Problem 4.10. Fix $p \in \mathbb{C}[Z_1, Z_2]$ to be the polynomial defined by

$$p(z_1, z_2) = a_{10} z_1 + a_{01} z_2 + a_{20} z_1^2 + a_{11} z_1 z_2 + a_{02} z_2^2.$$

Find necessary and sufficient conditions for the existence of a holomorphic function function q defined on the bi-disc $\mathbb{D}^2$ with $q^{(k)}(0) = 0$ for $|k| \le 2$, such that $\|p + q\|_{\mathbb{D}^2,\infty} \le 1$.

Let f be an analytic function on $\mathbb{D}^2$. Suppose f is represented by the power series

$$f(z) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$$

and $a_{00} = 0$. Also assume that f maps $\mathbb{D}^2$ into $\mathbb{D}$. This happens if and only if $\chi \circ f$ maps $\mathbb{D}^2$ to H_+, where

$$\chi \circ f(z) = (1 + f(z))(1 - f(z))^{-1} = 2\left(c_{00} + \sum_{m,n=1}^{\infty} c_{mn} z_1^m z_2^n\right),$$

$c_{00} = 1/2$ and the coefficients c_{mn} are from the Lemma 4.13. In this section, we set $c_{00} = 1/2$, wherever it occurs. If ϕ denotes the Korányi-Pukánszky function corresponding to the coefficients (c_{mn}), then ϕ is positive.

The matrix of ϕ: For a fixed $k \in \mathbb{Z}$, define $P_k := \{(x, y) \mid x + y = k\}$. The sequence (P_k) is a sequence of disjoint subsets of $\mathbb{Z}^2$. Besides

$$\bigsqcup_{k \in \mathbb{Z}} P_k = \mathbb{Z}^2.$$

An order on $\mathbb{Z}^2$, which we call the `D-slice` ordering, is defined below. Clearly, it is different from the usual co-lexicographic order. The matrix computations that follow are transparent because of the D-slice ordering that we use in describing the matrix of the Korányi-Pukánszky function ϕ.

Definition 4.11 (D-slice ordering)**.** Suppose $(x_1, y_1) \in P_l$ and $(x_2, y_2) \in P_m$ are two elements in $\mathbb{Z}^2$. Then

1. If $l = m$, then $(x_1, y_1) < (x_2, y_2)$ is determined by the lexicographic ordering on $P_l \subseteq \mathbb{Z}^2$ and
2. if $l < m$ (resp., if $l > m$), then $(x_1, y_1) < (x_2, y_2)$ (resp., $(x_1, y_1) > (x_2, y_2)$).

The following theorem describes the Korányi-Pukánszky function ϕ with respect to the D-slice ordering on $\mathbb{Z}^2$.

Theorem 4.12. *Let $(c_{mn})_{m,n\in\mathbb{N}_0}$ be an infinite array of complex numbers. The matrix of the Korányi-Pukánszky function ϕ in the D-slice ordering corresponding to this array is of the form*

$$\begin{array}{c c}
 & \begin{array}{ccccc} \cdots & P_{-1} & P_0 & P_1 & \cdots \end{array} \\
\begin{array}{c} \vdots \\ P_{-1} \\ P_0 \\ P_1 \\ \vdots \end{array} &
\begin{pmatrix}
 & \vdots & \vdots & \vdots & \\
\cdots & I & C_1^* & C_2^* & \cdots \\
\cdots & C_1 & I & C_1^* & \cdots \\
\cdots & C_2 & C_1 & I & \cdots \\
 & \vdots & \vdots & \vdots &
\end{pmatrix}
\end{array},$$

where $C_n := c_{n0} I + c_{n-1,1} B^* + \cdots + c_{0n} {B^*}^n$, $n \in \mathbb{N}$.

Proof. With respect to the D-slice ordering on $\mathbb{Z}^2$, the matrix corresponding to the function ϕ is a doubly infinite block matrix, where (k,n) element in (l,m) block, which is $\phi\left((k,-k+l)-(n,-n+m)\right)$. is computed as follows, separately, in three different cases:

First, let $k-n<0$.

The quantity $\phi((k,-k+l)-(n,-n+m))$ is non-zero only if $k-n \geq l-m$. Hence if $l \geq m$, then $\phi((k,-k+l)-(n,-n+m)) = 0$. Now, assume $l<m$. In this case, the possible values for $k-n$ are $l-m, l-m+1, \ldots, -1$, otherwise $\phi((k,-k+l)-(n,-n+m)) = 0$. For $p \in \{0,1,\ldots,-l+m-1\}$ and $k-n = l-m+p$, we have

$$\phi((k,-k+l)-(n,-n+m)) = \overline{c}_{m-l-p,p}.$$

Second, let $k-n=0$.

$$\phi(0,l-m)=\begin{cases} c_{0,l-m} & \text{if } l\geq m \\ \overline{c}_{0,m-l} & \text{if } l<m \end{cases}$$

Finally, let $k-n>0$.

The quantity $\phi((k,-k+l)-(n,-n+m))$ is non-zero only if $k-n\leq l-m$. Hence if $l\leq m$, then $\phi((k,-k+l)-(n,-n+m))=0$. Now, assume $l>m$. In this case the possible values for $k-n$ are $l-m, l-m-1,\dots,1$ otherwise $\phi((k,-k+l)-(n,-n+m))$. For $p\in\{0,1,\dots,l-m-1\}$ and $k-n=l-m-p$, we have

$$\phi((k,-k+l)-(n,-n+m))=c_{l-m-p,p}.$$

Therefore, the (l,m) block in the matrix of ϕ is given exactly by the following rule:

1. C^*_{m-l} if $l<m$,
2. C_{l-m} if $l>m$,
3. I if $m=l$.

Hence the matrix of the Korányi-Pukánszky function ϕ in the D-slice ordering corresponding to the array (c_{mn}) is of the form

$$\begin{array}{c} \\ \vdots \\ P_{-1} \\ P_0 \\ P_1 \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{ccccc} \cdots & P_{-1} & P_0 & P_1 & \cdots \end{array} \\
\begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & I & C_1^* & C_2^* & \cdots \\ \cdots & C_1 & I & C_1^* & \cdots \\ \cdots & C_2 & C_1 & I & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix}
\end{array}$$

□

Assume that the power series $\sum_{j,k=0}^{\infty} a_{jk} z_1^j z_2^k$, with $a_{00}=0$, represents a holomorphic function f defined on the bi-disc and that $\|f\|_{\mathbb{D}^2,\infty}\leq 1$. Let $2\big(c_{00}+\sum_{j,k=1}^{\infty} c_{jk} z_1^j z_2^k\big)$ be the power series representation for $\chi\circ f$ on the bi-disc $\mathbb{D}^2$ for some choice of complex number c_{mn} which are determined from the coefficients a_{mn} of the function f.

Lemma 4.13. *For all $n\in\mathbb{N}$, setting $A_n := a_{n0}I + a_{n-1,1}B^* + \cdots + a_{0n}B^{*n}$, $C_n = c_{n0}I + c_{n-1,1}B^* + \cdots + c_{0n}B^{*n}$, we have*

$$C_n = A_n + \sum_{j=1}^{n-1} A_j C_{n-j}.$$

Proof. Let $C(z_1, z_2) := \sum_{i,j=0}^{\infty} c_{ij} z_1^i z_2^j$. We have

$$1 + f(z_1, z_2) + f(z_1, z_2)^2 + \cdots = \frac{(\chi \circ f)(z_1, z_2)}{2} + c_{00} = C(z_1, z_2).$$

Thus $C(z_1, z_2)(1 - f(z_1, z_2)) = 1$, which is the same as

$$\begin{aligned} & (1 + c_{10} z_1 + c_{01} z_2 + c_{20} z_1^2 + c_{11} z_1 z_2 + c_{02} z_2^2 + \cdots) \times \\ & \quad (1 - a_{10} z_1 - a_{01} z_2 - a_{20} z_1^2 - a_{11} z_1 z_2 - a_{02} z_2^2 + \cdots) \\ = & \quad 1. \end{aligned}$$

Now comparing the coefficient of $z_1^{n-k} z_2^k$ we have

$$c_{n-k,k} = \sum_{p=0}^{k} \sum_{j=k}^{n} a_{n-j,p} c_{j-k,k-p},$$

where $a_{00} = 0$.
The coefficient of B^{*k} in $A_n + \sum_{i=1}^{n-1} A_i C_{n-i}$ is

$$\begin{aligned} & a_{n-k,k} c_{00} + a_{n-k,k-1} c_{01} + a_{n-k-1,k} c_{10} + a_{n-k,k-2} c_{02} \\ & \quad + a_{n-k-1,k-1} c_{11} + a_{n-k-2,k} c_{20} + \cdots \\ = & (a_{n-k,k} c_{00} + a_{n-k,k-1} c_{01} + \cdots + a_{n-k,0} c_{0k}) + \\ & \quad (a_{n-k-1,k} c_{10} + a_{n-k-1,k-1} c_{11} + \cdots + a_{n-k-1,0} c_{1,k}) + \cdots \\ & \quad \cdots + (a_{0k} c_{n-k,0} + a_{0,k-1} c_{n-k,1} + \cdots + a_{00} c_{n-k,k}) \\ = & \sum_{p=0}^{k} \sum_{j=k}^{n} a_{n-j,p} c_{j-k,k-p} \end{aligned}$$

completing the proof of the claim. □

In view of the Theorem 4.12 and the Lemma 4.13, the CF Problem 4.10 takes the following form:

Theorem 4.14. *For any polynomial p of the form*

$$p(z) = a_{10} z_1 + a_{01} z_2 + a_{20} z_1^2 + a_{11} z_1 z_2 + a_{02} z_2^2,$$

there exists a holomorphic function q, defined on the bi-disc $\mathbb{D}^2$, with $q^{(k)}(0) = 0$ for $|k| = 0, 1, 2$, such that

$$\|p + q\|_{\mathbb{D}^2, \infty} \leq 1$$

if and only if

$$\begin{pmatrix} I & C_1^* & C_2^* \\ C_1 & I & C_1^* \\ C_2 & C_1 & I \end{pmatrix}$$

is non-negative definite and for each $k \geq 3$, there exists $C_k = c_{k0}I + c_{k-1,1}B^* + \cdots + c_{0k}{B^*}^k$ such that the Korányi-Pukánszky function ϕ corresponding to $(c_{mn})_{m,n\in\mathbb{N}_0}$ is positive.

Lemma 4.15. *If A_n and C_n are as defined above, then*

$$\begin{pmatrix} I & C_1^* & C_2^* & \cdots & C_n^* \\ C_1 & I & C_1^* & \cdots & C_{n-1}^* \\ C_2 & C_1 & I & \cdots & C_{n-2}^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_n & C_{n-1} & C_{n-2} & \cdots & I \end{pmatrix} \geq 0$$

if and only if

$$\left\| \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_n \\ 0 & A_1 & A_2 & \cdots & A_{n-1} \\ 0 & 0 & A_1 & \cdots & A_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_1 \end{pmatrix} \right\| \leq 1.$$

Proof. For each $n \in \mathbb{N}$, C_n commutes with C_m and A_m for all $m \in \mathbb{N}$ and hence we can adapt the proof of the Lemma 4.7 to complete the proof in this case. □

Since the adjoint of the bilateral shift B^* on $\ell^2(\mathbb{Z})$ is unitarily equivalent to the multiplication operator M_z on $L^2(\mathbb{T})$, it follows that A_n and C_n are unitarily equivalent to the multiplication operators M_a and M_c respectively, where $a(z) = a_{n0} + a_{n-1,1}z + \cdots + a_{0n}z^n$, and $c(z) = c_{n0} + c_{n-1,1}z + \cdots + c_{0n}z^n$. Now the Theorem 4.14 takes the equivalent form given below, where for the polynomial p of the form $p(z) = a_{10}z_1 + a_{01}z_2 + a_{20}z_1^2 + a_{11}z_1z_2 + a_{02}z_2^2$, we have set

$$p_1(z) := a_{10} + a_{01}z \text{ and } p_2(z) = a_{20} + a_{11}z + a_{02}z^2. \tag{4.3}$$

Theorem 4.16. *For any polynomial p of the form*

$$p(z) = a_{10}z_1 + a_{01}z_2 + a_{20}z_1^2 + a_{11}z_1z_2 + a_{02}z_2^2,$$

there exists a holomorphic function q, defined on the bi-disc $\mathbb{D}^2$, with $q^{(k)}(0) = 0$ for $|k| = 0, 1, 2$, such that

$$\|p + q\|_{\mathbb{D}^2,\infty} \leq 1$$

if and only if $|p_2| \leq 1 - |p_1|^2$ and there exists a holomorphic function $f : \mathbb{D} \to \mathscr{B}(L^2(\mathbb{T}))$ with

$$\|f\|^{\rm op}_{\mathbb{D},\infty} \leq 1 \text{ and } \frac{f^{(k)}(0)}{k!} = M_{p_k} \text{ for all } k \geq 0,$$

where $p_0 = 0$ and for $k \geq 3$, $p_k \in \mathbb{C}[Z]$ is a polynomial of degree less than or equal to k. Here M_{p_k} is the multiplication operator on $L^2(\mathbb{T})$ induced by the polynomial p_k.

Thus the Problem 4.10 has been reduced to a one variable problem except it now involves holomorphic functions taking values in $\mathscr{B}(L^2(\mathbb{T}))$. To discuss this variant of the CF problem, the following definition will be useful.

Definition 4.17 (Completely Polynomially Extendible)**.** Suppose $k \in \mathbb{N}$ and $\{p_j\}_{j=1}^k$ is a sequence of polynomials, with $\deg(p_j) \leq j$ for all $j = 1, 2, \ldots, k$. Then $\mathcal{T}(M_{p_1}, \ldots, M_{p_k})$ will be called n-`polynomially extendible` if $\|\mathcal{T}(M_{p_1}, \ldots, M_{p_k})\| \leq 1$ and there exists a sequence of polynomials $\{p_l\}_{l=k+1}^n$, with $\deg(p_l) \leq l$, such that $\|\mathcal{T}(M_{p_1}, \ldots, M_{p_n})\| \leq 1$. Also, $\mathcal{T}(M_{p_1}, \ldots, M_{p_k})$ will be called `completely polynomially extendible` if the operator $\mathcal{T}(M_{p_1}, \ldots, M_{p_k})$ is n-polynomially extendible for all $n \in \mathbb{N}$.

For $p_1, p_2 \in \mathbb{C}[Z]$, polynomials of degree at most 1 and 2 respectively, let P denote the polynomial $P(z) = M_{p_1} z + M_{p_2} z^2$. We shall call P to be a polynomial in the CF class if given these polynomials p_1, p_2, there is a holomorphic function $f : \mathbb{D} \to \mathscr{B}(L^2(\mathbb{T}))$ satisfying properties stated in the Theorem 4.16. Such a function f is called a CF-extension of the polynomial P. It follows that a solution to the Problem 4.10 exists if and only if the polynomial P is in the CF class. We have therefore proved the following theorem.

Theorem 4.18. *A solution to the Problem 4.10 exists if and only if the corresponding one variable operator valued polynomial P is in the CF class. Or, equivalently, $\mathcal{T}(M_{p_1}, M_{p_2})$ is completely polynomially extendible.*

It is clear, from the Theorem 4.16, that $|p_1|^2 + |p_2| \leq 1$ is a necessary condition for the existence of a solution to the Problem 4.10. This condition via Parrott's theorem is also equivalent to the condition $\|\mathcal{T}(M_{p_1}, M_{p_2})\| \leq 1$.

We now give some instances, where this necessary condition is also sufficient for the existence of a solution to the Problem 4.10. This would amount to find condition for $\mathcal{T}(M_{p_1}, M_{p_2})$ to be completely polynomially extendible.

Theorem 4.19. *Let $p_1(z) = \gamma + \delta z$ and $p_2(z) = (\alpha + \beta z)(\gamma + \delta z)$ for some choice of complex numbers α, β, γ and δ. Assume that $|p_1|^2 + |p_2| \le 1$. If either $\alpha\beta\gamma\delta = 0$ or $\arg(\alpha) - \arg(\beta) = \arg(\gamma) - \arg(\delta)$, then $\mathcal{T}(M_{p_1}, M_{p_2})$ is completely polynomially extendible.*

Proof. All through this proof, for brevity of notation, we will let $\|f\|$ stand for the norm $\sup\{\|f(z)\|_{\text{op}} : z \in \mathbb{D}\}$, for any holomorphic function $f : \mathbb{D} \to \mathscr{B}(L^2(\mathbb{T}))$.

Case 1: Suppose $\beta = 0$. Then $P(z) = M_{p_1}(z + \alpha z^2)$. Let $p(z) = z + \alpha z^2$. Using Nehari's theorem, we extend p to the function $\tilde{p}(z) = z + \alpha z^2 + \alpha_3 z^3 + \cdots$ such that $\|\tilde{p}\|_{\mathbb{D},\infty} = \|\mathcal{T}(1,\alpha)\|$. Define $f(z) = M_{p_1}\tilde{p}(z) = M_{p_1}z + M_{p_2}z^2 + M_{p_3}z^3 + \cdots$, where $p_k = \alpha_k p_1$. Also,

$$\|f\| = \sup_{z\in\mathbb{D}} \|M_{p_1}\tilde{p}(z)\| = \|M_{p_1}\| \sup_{z\in\mathbb{D}} |\tilde{p}(z)| = \|M_{p_1}\|\|\mathcal{T}(1,\alpha)\|.$$

Thus $\|f\| = \|M_{p_1} \otimes \mathcal{T}(1,\alpha)\| = \|\mathcal{T}(M_{p_1}, M_{p_2})\| \le 1$. Hence f is a required CF-extension of P.

Case 2: Suppose $\alpha = 0$. Then, $P(z) = M_{p_1}(z + \beta M_z z^2)$. Let $Q(z) = z + \beta M_z z^2$ and $r(z_1, z_2) = z_1(1 + \beta z_2)$. Define $s(z_2) = 1 + \beta z_2$. Suppose $\tilde{s}(z_2) = s(z_2) + \beta_2 z_2^2 + \beta_3 z_2^3 + \cdots$ be such that $\|\tilde{s}\|_{\mathbb{D},\infty} = \|\mathcal{T}(1,\beta)\|$. If $\tilde{r} := z_1\tilde{s}(z_2)$, then $\|\tilde{r}\| = \|\tilde{s}\| = \|\mathcal{T}(1,\beta)\|$. If $\tilde{Q}(z) = z + M_{\beta z}z^2 + M_{\beta_2 z^2}z^2 + \cdots$ and $f(z) = M_{p_1}\tilde{Q}(z)$, then $\|f\| = \|M_{p_1}\tilde{Q}\| \le \|M_{p_1}\|\|\tilde{Q}\|$. Since $\tilde{s}(M_z) = \tilde{Q}(z)$, from the von-Neumann inequality it follows that $\|\tilde{Q}\| \le \|\tilde{s}\|$. Therefore, $\|f\| \le \|M_{p_1}\|\|\mathcal{T}(1,\beta)\| = \|\mathcal{T}(M_{p_1}, \beta M_{p_1})\|$. Hence

$$\|f\| \le \left\| \begin{pmatrix} M_z & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} M_{p_1} & \beta M_{p_1} \\ 0 & M_{p_1} \end{pmatrix} \begin{pmatrix} M_z^* & 0 \\ 0 & I \end{pmatrix} \right\| = \|\mathcal{T}(M_{p_1}, M_{p_2})\| \le 1.$$

Therefore f is a CF-extension of P.

Case 3: Suppose $\alpha \ne 0$ and $\beta \ne 0$. Then, $P(z) = M_{p_1}\left(z + M_{\alpha+\beta z}z^2\right)$. Let $Q(z) := z + M_{\alpha+\beta z}z^2$. Define $r(z_1, z_2) := z_1 + \alpha z_1^2 + \beta z_1 z_2 = z_1\left(1 + \alpha z_1 + \beta z_2\right)$. Let $\lambda := |\alpha|/|\beta|$ and $a := \lambda/(1+\lambda)$. Define $s(z_1, z_2) := 1 + \alpha z_1 + \beta z_2 = (a + \alpha z_1) + \left(1 - a + \beta z_2\right)$. If $h_1(z_1) := a + \alpha z_1$ and $h_2(z_2) := 1 - a + \beta z_2$, then there exist $\tilde{h}_1 = a + \alpha z_1 + \alpha_2 z_1^2 + \cdots$ and $\tilde{h}_2 = 1 - a + \beta z_2 + \beta_2 z_2^2 + \cdots$ with $\left\|\tilde{h}_1\right\| = \left\|\mathcal{T}(a,\alpha)\right\|$ and $\left\|\tilde{h}_2\right\| = \left\|\mathcal{T}(1-a,\beta)\right\|$. If

$$\tilde{r}(z_1, z_2) := z_1\left(\tilde{h}_1(z_1) + \tilde{h}_2(z_2)\right) = z_1 + \alpha z_1^2 + \beta z_1 z_2 + \alpha_2 z_1^3 + \beta_2 z_1 z_2^2 + \cdots,$$

then $\|\tilde{r}\| \le \|\tilde{h}_1\| + \|\tilde{h}_2\|$. Let $\tilde{Q}(z) = Iz + M_{\alpha+\beta z}z^2 + M_{\alpha_2+\beta_2 z^2}z^3 + \cdots$ and $f(z) = M_{p_1}\tilde{Q}(z)$ $= \sum_j M_{p_j}z^j$, where $p_{k+1}(z) = \left(\alpha_k + \beta_k z^k\right)p_1$ for all $k > 1$. Thus $\|f\| \le \|M_{p_1}\|\|\tilde{Q}\|$. Since $\tilde{Q}(z) = \tilde{r}(z, M_z)$, from the von-Neumann inequality, it follows that

$$\|f\| \le \|M_{p_1}\|\|\tilde{r}\| \le \|M_{p_1}\|\left(\|\tilde{h}_1\| + \|\tilde{h}_2\|\right).$$

As $\mathcal{T}(a,|\alpha|) = \lambda\mathcal{T}(1-a,|\beta|)$, therefore $\left(\|\tilde{h}_1\| + \|\tilde{h}_2\|\right) = \|\mathcal{T}\left(1,|\alpha|+|\beta|\right)\|$ and hence

$$\|f\| \le \|M_{p_1}\|\|\mathcal{T}\left(1,|\alpha|+|\beta|\right)\| = \left\|\mathcal{T}\left(\|p_1\|,(|\alpha|+|\beta|)\|p_1\|\right)\right\|.$$

subcase 1: Suppose $\gamma \neq 0$, $\delta \neq 0$ and $\arg(\alpha) - \arg(\beta) = \arg(\gamma) - \arg(\delta)$. Then

$$(|\alpha|+|\beta|)\|p_1\| = \|(\alpha+\beta)p_1\| = \|p_2\|.$$

Our hypothesis clearly implies that $\|p_2\| + \|p_1\|^2 \le 1$. Hence $\|f\| \le 1$.
subcase 2: Suppose $\gamma = 0$ or $\delta = 0$. Then

$$(|\alpha|+|\beta|)\|p_1\| = \|(\alpha+\beta)p_1\| = \|p_2\|.$$

As in subcase 1, here also $\|f\| \le 1$ can be inferred easily. □

Remark 4.20. *In Problem* (4.10)*, If either* $p_1 \equiv 0$ *or* $p_2 \equiv 0$*, and* $\|\mathcal{T}(M_{p_1}, M_{p_2})\| \le 1$*, then* $\|P\| \le 1$ *and hence* f *in the Theorem* (4.16) *can be taken to be* P *itself.*

Having verified that the necessary condition $\|\mathcal{T}(M_{p_1}, M_{p_2})\| \le 1$ is also sufficient for P to be in the CF class in several cases, we expected it to be sufficient in general. But unfortunately this is not the case. We give an example of a polynomial P for which $\|\mathcal{T}(M_{p_1}, M_{p_2})\| \le 1$ but P is not in the CF class.

An Example: Let $p_1(z) = 1/\sqrt{2}$ and $p_2(z) = z^2/2$. We show that $\mathcal{T}(M_{p_1}, M_{p_2})$ is not even 3−polynomially extendible.

It can easily be seen that $\|\mathcal{T}(M_{p_1}, M_{p_2})\| \le 1$. Now suppose there exists a polynomial p_3 of degree at most 3 such that $\|\mathcal{T}(M_{p_1}, M_{p_2}, M_{p_3})\| \le 1$. Then Parrott's theorem guarantees the existence of a contraction $V \in \mathcal{B}\left(L^2(\mathbb{T})\right)$ such that

$$M_{p_3} = \left(I - M_{|p_1|^2} - M_{p_2}\left(I - M_{|p_1|^2}\right)^{-1} M^*_{p_2}\right)V - M_{p_2}\left(I - M_{|p_1|^2}\right)^{-\frac{1}{2}} M^*_{p_1}\left(I - M_{|p_1|^2}\right)^{-\frac{1}{2}} M_{p_2}.$$

As we have $(1-|p_1|^2)^2 - |p_2|^2 \equiv 0$, therefore operator in the first bracket is zero and hence

$$p_3 = \frac{-p_2^2\overline{p_1}}{1-|p_1|^2} = \sqrt{2}z^4.$$

Thus p_3 is a polynomial of degree more than 3 which is a contradiction. Hence $\mathcal{T}(M_{p_1}, M_{p_2})$ is not even 3− polynomially extendible.

We close this section with an open question: What are the properties we must impose on the polynomials p_1 and p_2 in addition to the requirement $\|\mathcal{T}(M_{p_1}, M_{p_2})\| \le 1$ to ensure that P is in the CF class?

4.5 Korányi-Pukánszky Theorem as an Application of Spectral Theorem

In this section we show that Korányi-Pukánszky Theorem 4.1 is an application of the well known spectral theorem. For simplicity we consider $n = 2$.

Theorem 4.21 (Korányi-Pukánszky Theorem)**.** *If the power series $\sum_{\alpha\in\mathbb{N}_0^n} c_\alpha z^\alpha$ represents a holomorphic function g on the polydisc $\mathbb{D}^n$, then $\Re(g(z)) \geq 0$ for all $z \in \mathbb{D}^n$ if and only if the map $\phi : \mathbb{Z}^n \to \mathbb{C}$ defined by*

$$\phi(\alpha) = \begin{cases} 2\Re c_\alpha & \text{if } \alpha = 0 \\ c_\alpha & \text{if } \alpha > 0 \\ c_{-\alpha} & \text{if } \alpha < 0 \\ 0 & \text{otherwise} \end{cases}$$

is positive, that is, the $k \times k$ matrix $(\!(\phi(m_i - m_j))\!)$ is non-negative definite for every choice of $m_1, \ldots, m_k \in \mathbb{Z}^n$.

Proof. Suppose the power series $2\sum_{\alpha\in\mathbb{N}_0^n} c_\alpha z^\alpha$ represents a holomorphic function g defined on the bi-disc $\mathbb{D}^2$ with the property that $\Re(g(z)) \geq 0$ for all $z \in \mathbb{D}^2$(This extra factor 2 has been put in to the power series just to make previous computation matched). Without loss of generality, we assume that $g(0) = 1$. The function g maps $\mathbb{D}^2$ to the right half plane H_+ if and only if $f := \chi^{-1} \circ g$ maps $\mathbb{D}^2$ to the unit disc $\mathbb{D}$. Suppose $\sum_{j,k=0}^{\infty} a_{jk} z_1^j z_2^k$ represents the function f. Then, $a_{00} = 0$ and the array of coefficients $(\!(a_{jk})\!)$ and $(\!(c_{jk})\!)$ are related by the formula obtained in the Lemma 4.15. The operators $I \otimes B^*$ and $B^* \otimes B^*$ are commuting unitaries and they have $\mathbb{T}^2$ as their joint spectrum. Now, applying spectral theorem and maximum modulus principle, we get the following:

$$\|f(I \otimes B^*, B^* \otimes B^*)\| = \|f\|_{\mathbb{D}^2,\infty}. \tag{4.4}$$

Also, we note that

$$f(I \otimes B^*, B^* \otimes B^*) = A_1 \otimes B^* + A_2 \otimes {B^*}^2 + \cdots,$$

where $A_n := a_{n0} I + a_{n-1,1} B^* + \cdots + a_{0n} {B^*}^n$ and $C_n := c_{n0} I + c_{n-1,1} B^* + \cdots + c_{0n} {B^*}^n$ as in the Lemma 4.15. Since $\|f\|_{\mathbb{D}^2,\infty} \leq 1$, it follows from (4.4) that $\|\mathcal{T}(A_1, \ldots, A_n)\| \leq 1$ for all

$n \in \mathbb{N}$. Now, from the Lemma 4.15, we conclude that

$$\begin{pmatrix} I & C_1^* & C_2^* & \cdots & C_n^* \\ C_1 & I & C_1^* & \cdots & C_{n-1}^* \\ C_2 & C_1 & I & \cdots & C_{n-2}^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_n & C_{n-1} & C_{n-2} & \cdots & I \end{pmatrix} \tag{4.5}$$

is non-negative for all $n \in \mathbb{N}$. Hence from the Theorem 4.12, we get that the Korányi-Pukánszky function ϕ corresponding to the array $(\!(c_{jk})\!)$ is positive.

Conversely, suppose the Korányi-Pukánszky function ϕ corresponding to the array $(\!(c_{jk})\!)$ is positive, where c_{00} is assumed to be $1/2$. Then, from the Theorem 4.12, we get that operator in (4.5) is non-negative for all $n \in \mathbb{N}$. Thus, from the Lemma 4.15 and the equation (4.4), we conclude that $\|\chi \circ g\|_{\mathbb{D}^2,\infty} \leq 1$, where $g(z_1, z_2) = 2\sum_{m,n=0}^{\infty} c_{mn} z_1^m z_2^n$. This is so if and only if g maps $\mathbb{D}^2$ to the right half plane H_+. Hence the theorem is proved. $\square$

5 A generalization of Nehari's Theorem

For a closed subspace M and a point x in a Hilbert space $\mathbb{H}$, the distance of M from x is attained at $P(x)$, where P is the orthogonal projection of $\mathbb{H}$ onto M. Nehari considered a similar problem but in the space $L^\infty(\mathbb{T})$. He evaluated the distance of a function f in $L^\infty(\mathbb{T})$ from the closed subspace $H^\infty(\mathbb{T})$. Before stating Nehari's theorem we shall give some definitions.

5.1 The Hankel Operator

Let $H^2(\mathbb{T})$ denote the Hardy space, namely the closed subspace of $L^2(\mathbb{T})$:

$$H^2(\mathbb{T}) := \{f \in L^2(\mathbb{T}) \,|\, \hat{f}(-n) = 0, n \in \mathbb{N}\},$$

where $\hat{f}(-n)$ is the Fourier coefficient of f with respect to the standard orthonormal basis z^n, $n \in \mathbb{Z}$ and $z \in \mathbb{T}$, of $L^2(\mathbb{T})$. Let P_- denote the orthogonal projection of $L^2(\mathbb{T})$ onto $L^2(\mathbb{T}) \ominus H^2$.

Definition 5.1 (Multiplication Operator)**.** For $\phi \in L^\infty(\mathbb{T})$, we define the multiplication operator $M_\phi : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ by the rule $M_\phi(f) = \phi f$, where $(\phi f)(z) = \phi(z) f(z)$, $z \in \mathbb{T}$.

For any $\phi \in L^\infty(\mathbb{T})$ and $f \in L^2(\mathbb{T})$, it is easy to see that $\phi f \in L^2(\mathbb{T})$ and that M_ϕ is bounded. Indeed $\|M_\phi\| = \|\phi\|_\infty$ (cf. [You88, Theorem 13.14]).

Definition 5.2 (Hankel Operator)**.** For $\phi \in L^\infty(\mathbb{T})$, define the Hankel operator with symbol ϕ to be the operator $P_- \circ M_\phi|_{H^2}$ and denote it by H_ϕ.

We recall the well known theorem of Nehari.

Theorem 5.3 (Nehari)**.** *If $\phi \in L^\infty(\mathbb{T})$ and H_ϕ is the corresponding Hankel operator, then*

$$\inf\{\|\phi - g\|_{\mathbb{T},\infty} : g \in H^\infty(\mathbb{T})\} = \|H_\phi\|_{op}.$$

5.2 Nehari's theorem for $L^2(\mathbb{T}^2)$

In this section, we will give a possible generalization of Nehari's theorem for $L^2(\mathbb{T}^2)$. This generalization is most conveniently stated in terms of the D-slice ordering on $\mathbb{Z}^2$, which we now recall. For a fixed $k \in \mathbb{Z}$, define $P_k := \{(x,y) \mid x+y=k\}$. The sequence (P_k) is a sequence of disjoint subsets of $\mathbb{Z}^2$ and $\bigsqcup_{k\in\mathbb{Z}} P_k = \mathbb{Z}^2$. The D-slice ordering on $\mathbb{Z}^2$ is the ordering:

Suppose $(x_1, y_1) \in P_l$ and $(x_2, y_2) \in P_m$ are two elements in $\mathbb{Z}^2$. Then

1. If $l = m$, then $(x_1, y_1) < (x_2, y_2)$ is determined by the lexicographic ordering on $P_l \subseteq \mathbb{Z}^2$ and
2. if $l < m$ (respectively, if $l > m$), then $(x_1, y_1) < (x_2, y_2)$ (respectively, $(x_1, y_1) > (x_2, y_2)$).

Let $A_1 = \bigsqcup_{k\in\mathbb{N}_0} P_k$ and $A_2 = \bigsqcup_{k\in\mathbb{N}} P_{-k}$. Define

$$H_1 := \left\{ f := \sum_{(m,n)\in A_1} a_{m,n} z_1^m z_2^n \mid f \in L^\infty(\mathbb{T}^2) \right\},\ H_2 := \left\{ f := \sum_{(m,n)\in A_2} a_{m,n} z_1^m z_2^n \mid f \in L^\infty(\mathbb{T}^2) \right\}.$$

H_1 and H_2 are two closed and disjoint subspaces of $L^\infty(\mathbb{T}^2)$ satisfying $L^\infty(\mathbb{T}^2) = H_1 \oplus H_2$. Now the answer to the following question on $L^2(\mathbb{T}^2)$ would be a natural generalization of the Nehari's theorem.

Question 5.4. For $\phi \in L^\infty(\mathbb{T}^2)$, what is $\operatorname{dist}_\infty(\phi, H_1)$, the distance of ϕ from the subspace H_1?

5.3 The Hankel Matrix corresponding to ϕ

Any $\phi \in L^2(\mathbb{T}^2)$ can be written as

$$\phi(z_1, z_2) = \sum_{m,n\in\mathbb{Z}} a_{m,n} z_1^m z_2^n = \sum_{m,n\in A_1} a_{m,n} z_1^m z_2^n + \sum_{m,n\in A_2} a_{m,n} z_1^m z_2^n.$$

Suppose $z_2 = \lambda z_1$. Then

$$\phi(z_1, \lambda z_1) = \sum_{k\geq 0} \left(\sum_{m+n=k} a_{m,n} \lambda^n \right) z_1^k + \sum_{k<0} \left(\sum_{m+n=k} a_{m,n} \lambda^n \right) z_1^k.$$

Setting $f_k^\phi(\lambda) := \sum_{m+n=k} a_{m,n} \lambda^n$, we have

$$\phi(z_1, \lambda z_1) = \sum_{k\in\mathbb{Z}} f_k^\phi(\lambda) z_1^k.$$

In this way, $L^2(\mathbb{T}^2)$ is first identified with $L^2(\mathbb{T})\otimes L^2(\mathbb{T})$ and then a second time with $L^2(\mathbb{T})\otimes \ell^2(\mathbb{Z})$, the identifications in both cases being isometric. For any $\phi \in L^\infty(\mathbb{T}^2)$, define the multiplication operator $M_\phi : L^2(\mathbb{T})\otimes\ell^2(\mathbb{Z}) \to L^2(\mathbb{T})\otimes\ell^2(\mathbb{Z})$ as follows

$$M_\phi\left(\sum_{j\in\mathbb{Z}} g_j \otimes e_j\right) := \sum_{k\in\mathbb{Z}}\left(\sum_{q\in\mathbb{Z}} g_q f_{q+k}\right) e_k.$$

Lemma 5.5. *For any $\phi \in L^\infty(\mathbb{T}^2)$, we have $\|M_\phi\| = \|\phi\|_{\mathbb{T}^2,\infty}$.*

Proof. Let $\phi \in L^\infty(\mathbb{T}^2)$ be an arbitrary element. From what we have said above, it follows that $\phi(z,\lambda z) = \sum_{k\in\mathbb{Z}} f_k^\phi(\lambda) z^k$ for some f_k^ϕ in $L^2(\mathbb{T})$. The set of vectors $\{z^i\otimes e_j : (i,j)\in\mathbb{Z}^2\}$ is an orthonormal basis in $L^2(\mathbb{T})\otimes\ell^2(\mathbb{Z})$. The matrix of the operator M_ϕ with respect to this basis and the D-slice ordering on its index set is of the form

$$\begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & M_{f_{-1}^\phi} & M_{f_0^\phi} & M_{f_1^\phi} & \cdots \\ \cdots & M_{f_{-2}^\phi} & M_{f_{-1}^\phi} & M_{f_0^\phi} & \cdots \\ \cdots & M_{f_{-3}^\phi} & M_{f_{-2}^\phi} & M_{f_{-1}^\phi} & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix}.$$

We know that $\|\phi\|_{\mathbb{T}^2,\infty} = \sup_{\lambda\in\mathbb{T}}\sup_{z\in\mathbb{T}}\left|\sum_{k\in\mathbb{Z}} f_k^\phi(\lambda)z^k\right|$. Thus

$$\begin{aligned}\|\phi\|_{\mathbb{T}^2,\infty} &= \sup_{\lambda\in\mathbb{T}}\left\|\begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & f_{-1}^\phi(\lambda) & f_0^\phi(\lambda) & f_1^\phi(\lambda) & \cdots \\ \cdots & f_{-2}^\phi(\lambda) & f_{-1}^\phi(\lambda) & f_0^\phi(\lambda) & \cdots \\ \cdots & f_{-3}^\phi(\lambda) & f_{-2}^\phi(\lambda) & f_{-1}^\phi(\lambda) & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix}\right\| \\ &= \left\|\begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & M_{f_{-1}^\phi} & M_{f_0^\phi} & M_{f_1^\phi} & \cdots \\ \cdots & M_{f_{-2}^\phi} & M_{f_{-1}^\phi} & M_{f_0^\phi} & \cdots \\ \cdots & M_{f_{-3}^\phi} & M_{f_{-2}^\phi} & M_{f_{-1}^\phi} & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix}\right\|.\end{aligned}$$

Hence $\|\phi\|_{\mathbb{T}^2,\infty} = \|M_\phi\|$ completing the proof. □

The Hilbert space $\ell^2(\mathbb{N}_0)$ and the normed linear subspace

$$\left\{(\ldots,0,x_0,x_1,\ldots) : \sum_{i\geq 0}|x_i|^2 < \infty \text{ with } x_0 \text{ at the } 0^{th} \text{ position}\right\}$$

of $\ell^2(\mathbb{Z})$ are naturally isometrically isomorphic. Let $H := L^2(\mathbb{T}) \otimes \ell^2(\mathbb{N}_0)$. The space H is a closed subspace of $L^2(\mathbb{T}) \otimes \ell^2(\mathbb{Z})$. We define Hankel operator H_ϕ, with symbol ϕ, to be the operator $P_{H^\perp} \circ M_{\phi_{|H}}$. Writing down the matrix for H_ϕ with respect to the bases $\{z^i \otimes e_j : i \in \mathbb{Z}, j = 0,1,2,\dots\}$ and $\{z^i \otimes e_{-j} : i \in \mathbb{Z}, j = 1,2,\dots\}$ in the spaces H and $H^\perp$ respectively, we get

$$H_\phi = \begin{pmatrix} M_{f_{-1}^\phi} & M_{f_{-2}^\phi} & M_{f_{-3}^\phi} & \cdots \\ M_{f_{-2}^\phi} & M_{f_{-3}^\phi} & M_{f_{-4}^\phi} & \cdots \\ M_{f_{-3}^\phi} & M_{f_{-4}^\phi} & M_{f_{-5}^\phi} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

We call this the Hankel matrix with symbol ϕ.

Let $\mathbb{H}$ be a Hilbert space. For any $(T_n)_{n\in\mathbb{N}} \subset \mathscr{B}(\mathbb{H})$, define a operator $H(T_1, T_2, \dots)$ as follows:

$$H(T_1, T_2, \dots) = \begin{pmatrix} T_1 & T_2 & T_3 & \cdots \\ T_2 & T_3 & T_4 & \cdots \\ T_3 & T_4 & T_5 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Lemma 5.6. *For ϕ in $L^\infty(\mathbb{T}^2)$, we have $\|H_\phi\| \le \operatorname{dist}_\infty(\phi, H_1)$.*

Proof. From the definition of H_ϕ and the Lemma 5.5, it can easily be seen that

$$\|H_\phi\| = \|P_{H^\perp} \circ M_{\phi_{|H}}\| \le \|M_\phi\| = \|\phi\|_{\mathbb{T}^2,\infty}.$$

Thus $\|H_\phi\| \le \|\phi\|_{\mathbb{T}^2,\infty}$. From the matrix representation of H_ϕ it is clear that for any g in H_1, $H_{\phi-g} = H_\phi$. Hence $\|H_\phi\| = \|H_{\phi-g}\| \le \|\phi - g\|_{\mathbb{T}^2,\infty}$. Thus the proof of the lemma is complete. $\square$

For $n \in \mathbb{N}$, $a_0, a_1, \dots, a_{n-1} \in \mathbb{C}$ and $(b_m)_{m\in\mathbb{N}} \subset \mathbb{C}$, define the following operator

$$T_n\left((b_m), a_0, a_1, \dots, a_{n-1}\right) := \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ b_1 & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & & \vdots \\ b_{n-1} & b_{n-2} & \cdots & a_0 \\ \vdots & \vdots & & \vdots \end{pmatrix}.$$

Lemma 5.7. *Suppose $f_0, f_1, \dots, f_{n-1} \in L^\infty(\mathbb{T})$ and $(g_m) \subset L^\infty(\mathbb{T})$ are such that*

$$\sup_{\lambda\in\mathbb{T}} \|T_n\left((g_m(\lambda)), f_0(\lambda), \dots, f_{n-1}(\lambda)\right)\| \le 1.$$

Then, there exists $f_n \in L^\infty(\mathbb{T})$ such that

$$\sup_{\lambda \in \mathbb{T}} \| T_{n+1}\big((g_m(\lambda)), f_0(\lambda), \ldots, f_n(\lambda)\big) \| \le 1.$$

Proof. Let

$$Q(\lambda) = \begin{pmatrix} f_0(\lambda) & \cdots & f_{n-1}(\lambda) \end{pmatrix}, R(\lambda) = \begin{pmatrix} f_{n-1}(\lambda) & \cdots & f_0(\lambda) & g_1(\lambda) & \cdots \end{pmatrix}^{\mathrm{t}}$$

and

$$S(\lambda) = \begin{pmatrix} g_1(\lambda) & f_0(\lambda) & \cdots & f_{n-3}(\lambda) \\ g_2(\lambda) & g_1(\lambda) & \cdots & f_{n-4}(\lambda) \\ \vdots & \vdots & & \vdots \\ g_{n-1}(\lambda) & g_{n-2}(\lambda) & \cdots & g_1(\lambda) \\ \vdots & \vdots & & \vdots \end{pmatrix}.$$

All possible choices of $f_n(\lambda)$ for which $T_{n+1}\big((g_m(\lambda)), f_0(\lambda), \ldots, f_n(\lambda)\big)$ is a contraction are given, via Parrott's theorem (cf. [You88, Chapter 12, page 152]), by the formula

$$f_n(\lambda) = (I - ZZ^*)^{1/2} V (I - Y^* Y)^{1/2} - Z S(\lambda)^* Y, \tag{5.1}$$

where V is an arbitrary contraction and Y, Z are obtained from the formulae $R(\lambda) = (I - S(\lambda)S(\lambda)^*)^{1/2} Y$, $Q(\lambda) = Z(I - S(\lambda)^* S(\lambda))^{1/2}$.

Every entry of $I - S(\lambda)^* S(\lambda)$ is in L^∞ as function of λ. Thus all entries in $(I - S(\lambda)^* S(\lambda))^{1/2}$ are measurable functions which are essentially bounded. Consequently, so are entries of Z. A similar assertion can be made for Y. Therefore choosing $V = 0$ in equation (5.1), we get f_n with the required property. In fact, one can choose V to be any contraction whose entries are L^∞ functions. □

Theorem 5.8 (Nehari's theorem for $L^2(\mathbb{T}^2)$)**.** *If $\phi \in L^\infty(\mathbb{T}^2)$, then $\|H_\phi\| = \mathrm{dist}_\infty(\phi, H_1)$.*

Proof. From the Lemma 5.6, we know that $\|H_\phi\| \le \mathrm{dist}_\infty(\phi, H_1)$. Without loss of generality we assume that $\|H_\phi\| = 1$. Using the Lemma 5.7, we find $f_0^\phi \in L^\infty(\mathbb{T})$ such that the norm of $H\big(M_{f_0^\phi}, M_{f_{-1}^\phi}, \ldots\big)$ is at most 1. Repeated use of the Lemma 5.7 proves the theorem. □

5.4 CF problem in view of Nehari's theorem for $L^2(\mathbb{T})$

Fix $p \in \mathbb{C}[Z_1, Z_2]$ to be the polynomial defined by

$$p(z_1, z_2) = a_{10} z_1 + a_{01} z_2 + a_{20} z_1^2 + a_{11} z_1 z_2 + a_{02} z_2^2.$$

Denote $\phi(z_1, z_2) := \overline{z}_1^3 p(z_1, z_2) = a_{10}\overline{z}_1^2 + a_{01}\overline{z}_1^3 z_2 + a_{20}\overline{z}_1 + a_{11}\overline{z}_1^2 z_2 + a_{02}\overline{z}_1^3 z_2^2$. Suppose $p_1(\lambda) = a_{10} + a_{01}\lambda$ and $p_2(\lambda) = a_{20} + a_{11}\lambda + a_{02}\lambda^2$. Then, $\|H_\phi\| = \mathrm{dist}_\infty(\phi, H_1)$, where

$$H_\phi = \begin{pmatrix} M_{p_2} & M_{p_1} & 0 & \cdots \\ M_{p_1} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Thus, if there exists a holomorphic function $q : \mathbb{D}^2 \to \mathbb{C}$ with $q^{(k)}(0) = 0$ for $|k| \le 2$ such that $\|p + q\|_{\mathbb{D}^2,\infty} \le 1$, then $\|H_\phi\| \le \|p + q\|_{\mathbb{D}^2,\infty}$. Hence $\|H_\phi\| \le 1$ is a necessary condition for such a q to exist.

6 Operator Space Structures on $\ell^1(n)$

6.1 Operator space

Definition 6.1. An abstract operator space is a normed linear space V together with a norm $\|\cdot\|_k$ defined on the linear space

$$M_k(V) := \left\{\left(\!\left(v_{ij}\right)\!\right) | v_{ij} \in V, 1 \le i, j \le k\right\},\ k \in \mathbb{N},$$

with the understanding that $\|\cdot\|_1$ is the norm of V and the family of norms $\|\cdot\|_k$ satisfies the compatibility conditions:

1. $\|T \oplus S\|_{p+q} = \max\left\{\|T\|_p, \|S\|_q\right\}$ and

2. $\|ASB\|_q \le \|A\|_{op}\|S\|_p\|B\|_{op}$

for all $S \in M_q(V)$, $T \in M_p(V)$, $A \in M_{q\times p}(\mathbb{C})$ and $B \in M_{p\times q}(\mathbb{C})$.

Let $(V, \|\cdot\|_k)$ and $(W, \|\cdot\|_k)$ be two operator spaces. A linear bijection $T : V \to W$ is said to be a complete isometry if $T \otimes I_k : (M_k(V), \|\cdot\|_k) \to (M_k(W), \|\cdot\|_k)$ is an isometry for every $k \in \mathbb{N}$. Operator spaces $(V, \|\cdot\|_k)$ and $(W, \|\cdot\|_k)$ are said to be completely isometric if there is a linear complete isometry $T : V \to W$. A well known theorem of Ruan says that any operator space $(V, \|\cdot\|_k)$ can be embedded, completely isometrically, in to C^*-algebra $\mathscr{B}(\mathbb{H})$ for some Hilbert space $\mathbb{H}$. There are two natural operator space structures on any normed linear space V, which may coincide. These are the MIN and the MAX operator space structures defined below.

Definition 6.2 (MIN)**.** The MIN operator space structure denoted by MIN(V) on a normed linear space V is obtained by the isometric embedding of V in to the C^*-algebra $C((V^*)_1)$, the space of continuous functions on the unit ball $(V^*)_1$ of the dual space V^*. Thus for $\left(\!\left(v_{ij}\right)\!\right)$ in $M_k(V)$, we set

$$\left\|\left(\!\left(v_{ij}\right)\!\right)\right\|_{MIN} = \sup\left\{\left\|\left(\!\left(f(v_{ij})\right)\!\right)\right\| : f \in (V^*)_1\right\},$$

where the norm of a scalar matrix $\left(\!\left(f(v_{ij})\right)\!\right)$ is the operator norm in M_k.

Definition 6.3 (Max)**.** Let V be a normed linear space and $(\!(v_{ij})\!) \in M_k(V)$. Define

$$\|(\!(v_{ij})\!)\|_{MAX} = \sup\{\|(\!(Tv_{ij})\!)\| : T : V \to \mathscr{B}(\mathbb{H})\},$$

where the supremum is taken over all isometries T and all Hilbert spaces $\mathbb{H}$. This operator space structure is denoted by MAX(V).

These two operator space structures are extremal in the sense that for any normed linear space V, MIN(V) and MAX(V) are the smallest and largest operator space structures on V respectively. For any normed linear space V, Paulsen [Pau92] associates a very interesting constant, namely, $\alpha(V)$:

$$\alpha(V) := \sup\{\|I_V \otimes I_k\|_{(M_k(V),\|\cdot\|_{\text{MIN}})\to(M_k(V),\|\cdot\|_{\text{MAX}})} : k \in \mathbb{N}\}.$$

The constant $\alpha(V)$ is equal to 1 if and only if V has only one operator space structure on it. There are only a few examples of normed linear spaces for which $\alpha(V)$ is known to be 1. These include $\alpha(\ell^\infty(2)) = \alpha(\ell^1(2)) = 1$. In fact, it is known (cf. [Pis03, Page 79]) that $\alpha(V) > 1$ if $\dim(V) \geq 3$.

The map $\phi : \ell^\infty(n) \to \mathscr{B}(\mathbb{C}^n)$ defined by $\phi(z_1,\dots,z_n) = \text{diag}(z_1,\dots,z_n)$, is an isometric embedding of the normed linear space $\ell^\infty(n)$ in to the finite dimensional C*-algebra $\mathscr{B}(\mathbb{C}^n)$. Clearly, this is the MIN structure of the normed linear space $\ell^\infty(n)$. We shall, however prove that there is no such finite dimensional isometric embedding for the dual space $\ell^1(n)$. Never the less, we shall construct, explicitly, a number of possibly different isometric infinite dimensional embeddings of $\ell^1(n)$.

6.2 $\ell^1(n)$ has no isometric embedding into any M_k

In this section, we will show that there does not exist an isometric embedding of $\ell^1(n)$, $n > 1$, into any finite dimensional matrix algebra M_k, $k \in \mathbb{N}$. Without loss of generality, we prove this for the case of $n = 2$.

Lemma 6.4. *For $m \in \mathbb{N}$ and $\theta_1,\dots,\theta_m \in [0,2\pi)$, there exists $a_1, a_2 \in \mathbb{C}$ such that*

$$\max_{j=1,\dots,m} |a_1 + e^{i\theta_j} a_2| < |a_1| + |a_2|.$$

Proof. For any two non-zero complex numbers a_1, a_2, we have

$$\max_{j=1,\dots,m} |a_1 + e^{i\theta_j} a_2| = \max_{j=1,\dots,m} ||a_1| + e^{i\theta_j+\phi_2-\phi_1}|a_2||,$$

where ϕ_1 and ϕ_2 are the arguments of a_1 and a_2 respectively. Setting $\alpha_j = \theta_j + \phi_2 - \phi_1$, we have

$$\begin{aligned}\max_{j=1,\ldots,m} \left|a_1 + e^{i\theta_j} a_2\right|^2 &= \max_{j=1,\ldots,m} \left||a_1| + e^{i\alpha_j}|a_2|\right|^2 \\ &= \max_{j=1,\ldots,m} \left||a_1|^2|a_2|^2 + 2|a_1 a_2| cos \alpha_j\right|.\end{aligned}$$

Therefore

$$\max_{j=1,\ldots,m} \left|a_1 + e^{i\theta_j} a_2\right| = \left|a_1\right| + \left|a_2\right|$$

if and only if $\cos\alpha_j = 1$ for some j, that is, if and only if $\alpha_j = 0$ for some j. Choose a_1 and a_2 such that $\phi_1 - \phi_2 \neq \theta_j$ for all $j = 1, \ldots, m$. The existence of such a pair a_1 and a_2 proves the lemma. □

Theorem 6.5. *There is no isometric embedding of $\ell^1(2)$ into M_n for any $n \in \mathbb{N}$..*

Proof. Suppose there is a $n - dimensional$ isometric embedding ϕ of $\ell^1(2)$. Then this embedding ϕ is induced by a pair of operators $T_1, T_2 \in M_n$ of norm 1, defined by the rule $\phi(a_1, a_2) = a_1 T_1 + a_2 T_2$. Let U_1 and U_2 in M_{2n} be the pair of unitaries:

$$U_i := \begin{pmatrix} T_i & D_{T_i^*} \\ D_{T_i} & -T_i^*, \end{pmatrix} i = 1, 2,$$

where D_{T_i} is the positive square root of the (positive) operator $I - T_i^* T_i$. Now, we have

$$P_{\mathbb{C}^n}(a_1 U_1 + a_2 U_2)_{|\mathbb{C}^n} = a_1 T_1 + a_2 T_2.$$

(This dilating pair of unitaries is not necessarily commuting nor is it a power dilation!) Thus $\psi : \ell^1(2) \to M_{2n}(\mathbb{C})$ defined by $\psi(a_1, a_2) = a_1 U_1 + a_2 U_2$ is also an isometry. Since norms are preserved under unitary operations, without loss of generality, we assume $U_1 = I$ and U_2 to be a diagonal unitary, say, D. Let $D = \text{diag}\left(e^{i\theta_1}, \ldots, e^{i\theta_{2n}}\right)$. Now applying the Lemma 6.4, we obtain complex numbers a_1 and a_2 such that

$$\max_{j=1\ldots 2n} \left|a_1 + e^{i\theta_j} a_2\right| < \left|a_1\right| + \left|a_2\right|.$$

Hence ψ cannot be an isometry contradicting the hypothesis that ϕ is an isometry. □

Remark 6.6. *An amusing corollary of this theorem is that the two spaces $\ell^\infty(n)$ and $\ell^1(n)$ cannot be isometrically isomorphic for $n > 1$.*

6.3 Infinite dimensional embeddings of $\ell^1(n)$

Let $\mathbb{H}_1,\ldots,\mathbb{H}_n$ be Hilbert spaces and T_i be a contraction on $\mathbb{H}_i$ for $i=1,\ldots,n$. Assume that the unit circle $\mathbb{T}$ is contained in $\sigma(T_i)$, the spectrum of T_i, for $i=1,\ldots,n$. Denote

$$\tilde{T}_1 = T_1 \otimes I^{\otimes(n-1)}, \tilde{T}_2 = I \otimes T_2 \otimes I^{\otimes(n-2)}, \ldots, \tilde{T}_n = I^{\otimes(n-1)} \otimes T_n.$$

Theorem 6.7. *Suppose the operators $\tilde{T}_1,\ldots,\tilde{T}_n$ are defined as above. Then, the function*

$$f : \ell^1(n) \to \mathscr{B}(\mathbb{H}_1 \otimes \cdots \otimes \mathbb{H}_n)$$

defined by

$$f(a_1, a_2, \ldots, a_n) := a_1\tilde{T}_1 + a_2\tilde{T}_2 + \cdots + a_n\tilde{T}_n.$$

is an isometry.

Proof. Since $\mathbb{T} \subset \sigma(T_i)$ and T_i is a contraction for $i=1,\ldots,n$, it follows that $\mathbb{T} \subset \partial\sigma(T_i)$ for $i=1,\ldots,n$. From (cf. [Con90, Proposition 6.7, page 210]), we have $\mathbb{T} \subset \sigma_a(T_i)$ (approximate point spectrum of T_i) for $i=1,\ldots,n$. Thus for any $i \in \{1,\ldots,n\}$ and $\lambda \in \mathbb{T}$, there exists a sequence of unit vectors $(x_m^i)_{m\in\mathbb{N}}$ in $\mathbb{H}_i$ such that

$$\|(T_i - \lambda)(x_m^i)\| \longrightarrow 0 \text{ as } m \longrightarrow \infty.$$

Now, applying the Cauchy-Schwarz's inequality, we have

$$\begin{aligned} |\langle (T_i-\lambda)(x_m^i), (x_m^i)\rangle| &\le \|(T_i-\lambda)(x_m^i)\|\,\|(x_m^i)\| \\ &= \|(T_i-\lambda)(x_m^i)\| \longrightarrow 0. \end{aligned}$$

as $m \longrightarrow \infty$. Hence $\langle T_i(x_m^i), (x_m^i)\rangle \longrightarrow \lambda$ as $m \longrightarrow \infty$. Let $(a_1,\ldots,a_n)$ be any vector in $\ell^1(n)$ such that none of its co-ordinates zero. Let $\lambda_1 = e^{-i\arg(a_1)}, \lambda_2 = e^{-i\arg(a_2)}, \ldots, \lambda_n = e^{-i\arg(a_n)}$. Now for each $j \in \{1,\ldots,n\}$, we have $(x_m^j)_{m\in\mathbb{N}}$, a sequence of unit vectors from $\mathbb{H}_j$, such that

$$\langle T_j(x_m^j), (x_m^j)\rangle \longrightarrow \lambda_j \text{ as } m \longrightarrow \infty.$$

As m goes to ∞, we have

$$\begin{aligned} &\left|\langle (a_1 T_1 \otimes I^{\otimes(n-1)} + \cdots + a_n I^{\otimes(n-1)} \otimes T_n)(x_m^1 \otimes \cdots \otimes x_m^n), (x_m^1 \otimes \cdots \otimes x_m^n)\rangle\right| \\ &= \left|a_1\langle T_1(x_m^1), (x_m^1))\rangle + \cdots + a_n\langle T_n(x_m^n), (x_m^n))\rangle\right| \longrightarrow \left|a_1\lambda_1 + \cdots + a_n\lambda_n\right| \\ &= |a_1| + \cdots + |a_n| = \|(a_1,\ldots,a_n)\|_1. \end{aligned}$$

Hence $\|a_1\tilde{T}_1 + a_2\tilde{T}_2 + \cdots + a_n\tilde{T}_n\| \geq \|(a_1, a_2, \ldots, a_n)\|_1$. Also

$$\|a_1\tilde{T}_1 + a_2\tilde{T}_2 + \cdots + a_n\tilde{T}_n\| \leq |a_1|\|T_1\| + |a_2|\|T_2\| + \cdots + |a_n|\|T_n\|.$$

Hence $\|a_1\tilde{T}_1 + a_2\tilde{T}_2 + \cdots + a_n\tilde{T}_n\| = \|(a_1, a_2, \ldots, a_n)\|_1$, proving that f is an isometry.

If some of the co-ordinates in the vector $(a_1, \ldots, a_n)$ are zero, the same argument, as above, remains valid after dropping those co-ordinates. □

An adaptation of the technique involved in the proof of the Theorem 6.7, also proves the following theorem.

Theorem 6.8. *For $i = 1, \ldots, n$, let T_i be a contraction on a Hilbert space $\mathbb{H}_i$ and $\mathbb{T} \subseteq \sigma(T_i)$. Denote $\tilde{T}_i = T_1 \otimes \cdots \otimes T_i \otimes I_{\mathbb{H}_{i+1}} \otimes \cdots \otimes I_{\mathbb{H}_n}$. Then, the function*

$$f : \ell^1(n) \to \mathscr{B}(\mathbb{H}_1 \otimes \cdots \otimes \mathbb{H}_n)$$

defined by

$$f(a_1, a_2, \ldots, a_n) := a_1\tilde{T}_1 + a_2\tilde{T}_2 + \cdots + a_n\tilde{T}_n.$$

is an isometry.

Remark 6.9. *We have already noted that $\alpha(\ell^1(2)) = 1$. Therefore all the operator space structures on $\ell^1(2)$, defined in the Theorem 6.7, must be completely isometric to the MIN operator space structure.*

Suppose T_1 and T_2 are contractions on Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$ respectively with the property that $\mathbb{T} \subseteq \sigma(T_i)$ for $i = 1, 2$. Denote $\tilde{T}_1 = T_1 \otimes I_{\mathbb{H}_2}$ and $\tilde{T}_2 = I_{\mathbb{H}_1} \otimes T_2$. Then the map f defined as in the Theorem 6.7 is an isometry. The dilation theorem due to Sz.-Nagy (cf. [Pau02, Theorem 1.1, page 7]), gives unitaries $U_1 : \mathbb{K}_1 \to \mathbb{K}_1$ and $U_2 : \mathbb{K}_2 \to \mathbb{K}_2$ dilating the contractions T_1 and T_2 respectively. The operator space structure defined by the isometry $g : \ell^1(2) \to \mathscr{B}(\mathbb{K}_1 \otimes \mathbb{K}_2)$, where $g(a_1, a_2) = a_1U_1 \otimes I_{\mathbb{K}_2} + a_2I_{\mathbb{K}_1} \otimes U_2$, is no lesser than that of f. Since U_1 is a unitary map, it follows that the map $a_1U_1 \otimes I_{\mathbb{K}_2} + a_2I_{\mathbb{K}_1} \otimes U_2 \mapsto a_1I_{\mathbb{K}_1} \otimes I_{\mathbb{K}_2} + a_2U_1^* \otimes U_2$ is a complete isometry. Therefore, without loss of generality, for all operator space structures, defined in the Theorem 6.7, we can assume that T_1 is $I_{\mathbb{H}_1}$. Now suppose $k \in \mathbb{N}$ and $A_1, A_2 \in M_k$. The von-Neumann inequality implies that

$$\|A_1 \otimes I_{\mathbb{H}_1} \otimes I_{\mathbb{H}_2} + A_2 \otimes I_{\mathbb{H}_1} \otimes T_2\| \leq \|A_1 + A_2 z\|^{\mathrm{op}}_{\mathbb{D},\infty} = \|A_1 \otimes (1,0) + A_2 \otimes (0,1)\|_{MIN}.$$

Since MIN is the smallest operator space structure, it follows that all operator space structures on $\ell^1(2)$, defined in the Theorem 6.7 are completely isometric to the MIN structure.

Remark 6.10. *Here we note that all the operator space structures on $\ell^1(3)$, defined in the Theorem 6.7, are completely isometric to the MIN structure. Suppose T_1, T_2 and T_3 are contractions on Hilbert spaces $\mathbb{H}_1$, $\mathbb{H}_2$ and $\mathbb{H}_3$ respectively, with the property that $\mathbb{T} \subseteq \sigma(T_i)$, for $i = 1,2,3$. Then the map f defined as in the Theorem 6.7 is an isometry. Using the same arguments as in the Remark 6.9, here also we can assume that $T_1 = I_{\mathbb{H}_1}$. Let $k \in \mathbb{N}$ and $A_1, A_2, A_3 \in M_k$. Since $U_1^* \otimes U_2 \otimes I_{\mathbb{K}_3}$ and $U_1^* \otimes I_{\mathbb{K}_2} \otimes U_3$ commute, therefore via Ando's theorem, we conclude that*

$$\|A_1 \otimes I_{\mathbb{H}_1 \otimes \mathbb{H}_2 \otimes \mathbb{H}_3} + A_2 \otimes I_{\mathbb{H}_1} \otimes T_2 \otimes I_{\mathbb{H}_3} + A_3 \otimes I_{\mathbb{H}_1} \otimes I_{\mathbb{H}_2} \otimes T_3\| \le \|A_1 + A_2 z_2 + A_3 z_3\|^{\mathrm{op}}_{\mathbb{D}^2,\infty}.$$

The right hand quantity in this inequality is $\|A_1 \otimes (1,0,0) + A_2 \otimes (0,1,0) + A_3 \otimes (0,0,1)\|_{MIN}$. Since MIN is the smallest operator space structure, therefore all the operator space structure on $\ell^1(3)$, defined in the Theorem 6.7, are completely isometric to the MIN structure.

6.4 Operator space structures on $\ell^1(n)$ different from the MIN structure

Due to Parrott's example [Par70], it is known that a linear contractive map on $\ell^1(3)$ may not be completely contractive. An explicit example for this is give also in the paper of G. Misra [Mis94]. This example explains that there are more than one operator space structure on $\ell^1(3)$. In this section, using the example in [Mis94], we give an explicit operator space structure on $\ell^1(3)$, which is different from the MIN structure.

Consider the following 2×2 unitary operators:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, U := \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \text{ and } V := \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

It is clear that the map $h : \ell^1(3) \to M_2$, defined by $h(z_1, z_2, z_3) = z_1 I + z_2 U + z_3 V$, is of norm at most 1. The computations done in [Mis94] includes the following:

$$\|I \otimes I + U \otimes U + V \otimes V\| = 3. \tag{6.1}$$

and

$$\sup_{z_1, z_2, z_3 \in \mathbb{D}} \|z_1 I + z_2 U + z_3 V\| < 3. \tag{6.2}$$

Choose a diagonal operator D on $\ell^2(\mathbb{Z})$ such that $\|D\| \le 1$ and $\mathbb{T} \subset \sigma(D)$. Define

$$\tilde{T}_1 := \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}, \tilde{T}_2 := \begin{bmatrix} U & 0 \\ 0 & D \end{bmatrix}, \tilde{T}_3 := \begin{bmatrix} V & 0 \\ 0 & D \end{bmatrix}$$

and

$$\hat{T}_1 = \tilde{T}_1 \otimes I \otimes I,\ \hat{T}_2 = I \otimes \tilde{T}_2 \otimes I,\ \hat{T}_3 = I \otimes I \otimes \tilde{T}_n.$$

Let

$$S_1 := \hat{T}_1 \oplus I,\ S_2 := \hat{T}_2 \oplus U,\ S_3 := \hat{T}_3 \oplus V$$

be operators on a Hilbert space $\mathbb{K}$.

Define

$$S : (\ell^1(3), MIN) \longrightarrow B(\mathbb{K})$$

by

$$S(e_1) = S_1,\ S(e_2) = S_2,\ S(e_3) = S_3$$

and extend it linearly.

From the Theorem 6.7, we know that the function $(z_1, z_2, z_3) \mapsto z_1\hat{T}_1 + z_2\hat{T}_2 + z_3\hat{T}_3$ is an isometry and since h is of norm at most 1, it follows that the map $(z_1, z_2, z_3) \mapsto z_1 S_1 + z_2 S_2 + z_3 S_3$ is also an isometry. Consequently, there is an operator space structure on $\ell^1(3)$ for which S is a complete isometry. Also from (6.1), we have

$$\|S_1 \otimes I + S_2 \otimes U + S_3 \otimes V\| \geq \|I \otimes I + U \otimes U + V \otimes V\| = 3.$$

On the other hand from (6.2), we have

$$\sup_{z_1, z_2, z_3 \in \mathbb{D}} \|z_1 I + z_2 U + z_3 V\| < 3$$

and hence the operator space structure induced by S is different from the MIN structure.

List of Symbols

$\mathbb{N}$	The set of all positive integers
$\mathbb{Z}$	The set of all integers
$\mathbb{N}_0$	The set of all non-negative integers
$\mathbb{C}$	Complex plane
$\mathbb{C}[Z_1,\dots,Z_m]$	The set of all polynomials in m variables
$z\cdot x$	$\sum\limits_{j=1}^{l} z_j x_j$ for $x=(x_1,\dots,x_l)\in B^l,\ z=(z_1,\dots,z_l)\in\mathbb{C}^l$
$\\|f\\|_{\Omega,\infty}$	$\sup\{\|f(z)\|: z\in\Omega\}$
$\mathbb{H}$	A separable Hilbert space
$\mathscr{B}(\mathbb{H})$	The set of all bounded operators on $\mathbb{H}$
$\sigma(T)$	Spectrum of T
$H^\infty(\Omega)$	The set of all bounded holomorphic functions on Ω
$\mathbb{D}$	unit disk in $\mathbb{C}$
$\mathbb{T}$	Circle of unit length
$\\|p\\|_{\Omega,\infty}^{\mathrm{op}}$	$\sup\{\\|p(z)\\|_{op}: z\in\Omega\}$
$K_G^{\mathbb{C}}$	Complex Grothendieck Constant
$\mathscr{P}_k[Z_1,\dots,Z_n]$	The set of all polynomials of degree k in n variables
$H^\infty(\Omega)$	The set of all complex valued bounded holomorphic function on Ω
$H^\infty(\Omega,\mathbb{D})$	$\{f\in H^\infty(\Omega): \\|f\\|_{\Omega,\infty}\le 1\}$
$H^\infty_\omega(\Omega,\mathbb{D})$	$\{f\in H^\infty(\Omega,\mathbb{D}): f(\omega)=0\}$
$Df(\omega)$	$\left(\frac{\partial}{\partial z_1}f(\omega),\dots,\frac{\partial}{\partial z_m}f(\omega)\right)$
$\mathscr{L}[Z_1,\dots,Z_n]$	$\{a_1z_1+\cdots+a_nz_n: a_i\in\mathbb{C}, i=1,\dots,n\}$
$[x^\sharp,y]$	$\sum_j x_j y_j$
$x^\sharp$	$\mathbb{C}$ – valued linear map on $\mathbb{H}$ defined by $x^\sharp(y)=[x^\sharp,y]$
$\mathbb{H}^\sharp$	$\{x^\sharp: x\in\mathbb{H}\}$
$L^2(\mathbb{T})$	The set of all square integrable functions on $\mathbb{T}$ with respect to

$H^2(\mathbb{T})$	Hardy space
$L^\infty(\mathbb{T}^n)$	The set of all essentially bounded functions $\mathbb{T}^n$ with respect to Lebsegue measure and equipped with the essential sup norm
M_ϕ	Multiplication operator corresponding to ϕ
H_ϕ	Hankel matrix corresponding to symbol ϕ
$\\|\boldsymbol{T}\\|_\infty$	$\max\{\\|T_1\\|,\ldots,\\|T_n\\|\}$
$C_k(n)$	$\sup\{\\|p(\boldsymbol{T})\\| : \\|p\\|_{\mathbb{D}^n,\infty}\le 1, p$ is of degree at most k, $\\|\boldsymbol{T}\\|_\infty \le 1\}$
$C(n)$	$\lim_{k\to\infty} C_k(n)$
$\mathfrak{T}(A_1,\ldots,A_n)$	$\begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_n \\ 0 & A_1 & A_2 & \cdots & A_{n-1} \\ 0 & 0 & A_1 & \cdots & A_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_1 \end{pmatrix}$
$M_k(B)$	The set of all $k\times k$ matrices with entries in Banach space B
$\mathrm{dist}_\infty(\phi,K)$	Distance of ϕ from $K$with respect to essential sup norm
A_1	$\bigsqcup_{k\in\mathbb{N}_0} P_k$
A_2	$\bigsqcup_{k\in\mathbb{N}} P_{-k}$
H_1	$\left\{f := \sum\limits_{(m,n)\in A_1} a_{m,n} z_1^m z_2^n \mid f\in L^\infty(\mathbb{T}^2)\right\}$
H_2	$\left\{f := \sum\limits_{(m,n)\in A_2} a_{m,n} z_1^m z_2^n \mid f\in L^\infty(\mathbb{T}^2)\right\}$
$\ell^2(\mathbb{N}_0)$	$\left\{(a_0,a_1,\ldots) : a_j\in\mathbb{C}\text{ for all } j\in\mathbb{N}_0, \sum\limits_{j\ge 0} \|a_j\|^2 < \infty\right\}$
$p_{\scriptscriptstyle A}$	$\sum_{i,j=1}^n a_{ij} z_i w_j$, where $A = (\!(a_{ij})\!)$
$\triangle$	$\{(z_1,\ldots,z_n,z_1,\ldots,z_n) : \|z_i\| < 1, 1\le i\le n\}$
$p_{\scriptscriptstyle A,\triangle}$	the restriction of $p_{\scriptscriptstyle A}$ to the diagonal set $\triangle$
$\\|(z_1,\ldots,z_n)\\|_p$	$(\|z_1\|^p + \cdots + \|z_n\|^p)^{1/p}$
$\ell^p(n)$	$(\mathbb{C}^n, \\|.\\|_p)$
$\\|A\\|_{\ell^\infty(n)\to\ell^1(n)}$	Operator norm of $A : \ell^\infty(n)\to\ell^1(n)$
A_{xy}	The matrix $([x_j^\sharp, y_k])_{m\times m}$
$\mathscr{D}_\Omega^{(\omega)}$	$\left\{\left(\frac{1}{2}D^2 f(\omega), Df(\omega)\right) : f:\Omega\to\mathbb{D}\text{ is a analytic map with } f(\omega)=0\right\}$
M_m^s	The set of all $m\times m$ complex symmetric matrices
$\\|\cdot\\|_{\mathscr{D}}$	The norm in $M_m^s\times\mathbb{C}^m$ corresponding to the unit ball $\mathscr{D}_\Omega^{(\omega)}$
$\mathbb{U}$	$\{(z,v) : z\in\mathbb{C}, v\in\mathbb{H}\text{ such that } \|z\| + \\|v\\|^2 \le 1\}$
$\\|\cdot\\|_{\mathbb{U}}$	The norm in $\mathbb{C}\oplus\mathbb{H}$ corresponding to the unit ball $\mathbb{U}$
$\mathscr{P}_k(\Omega,E)$	$\{p\in\mathbb{C}[Z_1,\ldots,Z_n] : \deg(p)\le k\text{ and } p(\Omega)\subset E\}$.
$\mathscr{P}_k^\omega(\Omega,E)$	The set of all polynomials $p\in\mathscr{P}_k(\Omega,E)$ with $p(\omega)=0$.

M_k	The set of all $k \times k$ complex matrices
$(X)_1$	Open Unit Ball of Banach Space X
$\mathscr{P}(\mathbb{C}^m, M_k)$	The set of all M_k valued polynomials in m variables
$\mathscr{P}_k(\mathbb{C}^m, M_k)$	$\{p \in \mathscr{P}(\mathbb{C}^m, M_k) : \deg(p) \leq k\}$.
$\mathscr{P}_k^{(\omega)}(\mathbb{C}^m, M_k)$	The set of all polynomials $p \in \mathscr{P}_k(\mathbb{C}^m, M_k)$ such that $p(\omega) = 0$
$\mathscr{P}_n^{(\omega)}(\Omega, (M_k)_1)$	$\left\{p \in \mathscr{P}_n^{(\omega)}(\mathbb{C}^m, M_k) : \\|p\\|_{\Omega,\infty}^{\mathrm{op}} \leq 1\right\}$
B^*	Adjoint of the bilateral shift
$\ell^2(\mathbb{Z})$	$\left\{(\ldots, a_{-1}, a_0, a_1, \ldots) : a_j \in \mathbb{C} \text{ for all } j \in \mathbb{Z}, \sum_{j \in \mathbb{Z}} \|a_j\|^2 < \infty\right\}$
$C^*(a)$	C^*-algebra geneated by $1, a$ and a^*
$\sigma(a)$	$\{\lambda \in \mathbb{C} : a - \lambda 1$ is not invertible$\}$
$C(X)$	The set of all complex valued continuous functions on X
I_k	Identity operator on $\mathbb{C}^k$
I	Identity operator on $\mathbb{H}$
$\Re(z)$	Real part of z
H_+	The right half plane
P_k	$\{(x, y) \in \mathbb{Z}^2 : x + y = k\}$
	normalized Lebsegue measure and equipped with the L^2 norm
$arg(\alpha)$	Argument of the complex number α
$H(T_1, T_2, \ldots)$	$\begin{pmatrix} T_1 & T_2 & T_3 & \cdots \\ T_2 & T_3 & T_4 & \cdots \\ T_3 & T_4 & T_5 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$
$T_n((b_m), a_0, a_1, \ldots, a_{n-1})$	$\begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ b_1 & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & & \vdots \\ b_{n-1} & b_{n-2} & \cdots & a_0 \\ \vdots & \vdots & & \vdots \end{pmatrix}$
$M_k(V)$	$M_k \otimes V$
$MIN(V)$	MIN operator space structure on V
$MAX(V)$	MAX operator space structure on V
$\alpha(V)$	$\left\{\frac{\\|(v_{ij})\\|_{MAX}}{\\|v_{ij}\\|_{MIN}} : (v_{ij}) \in M_{k,l}(V), k \text{ and } l \text{ are arbitrary positive integers}\right\}$
$\dim(V)$	Dimension of the vector space V
D_T	The positive square root of the operator $I - T^*T$

Bibliography

[Agl90] Jim Agler, *Operator theory and the Carathéodory metric*, Invent. Math. **101** (1990), no. 2, 483–500. MR 1062972 (91f:47005)

[And63] T. Andô, *On a pair of commutative contractions*, Acta Sci. Math. (Szeged) **24** (1963), 88–90. MR 0155193 (27 #5132)

[Arv69] William B. Arveson, *Subalgebras of C^*-algebras*, Acta Math. **123** (1969), 141–224. MR 0253059 (40 #6274)

[Arv72] William Arveson, *Subalgebras of C^*-algebras. II*, Acta Math. **128** (1972), no. 3-4, 271–308. MR 0394232 (52 #15035)

[BW11] Mihály Bakonyi and Hugo J. Woerdeman, *Matrix completions, moments, and sums of Hermitian squares*, Princeton University Press, Princeton, NJ, 2011. MR 2807419 (2012d:47003)

[Con90] John B. Conway, *A course in functional analysis*, second ed., Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990. MR 1070713 (91e:46001)

[DMP68] R. G. Douglas, P. S. Muhly, and Carl Pearcy, *Lifting commuting operators*, Michigan Math. J. **15** (1968), 385–395. MR 0236752 (38 #5046)

[EPP00] Jörg Eschmeier, Linda Patton, and Mihai Putinar, *Carathéodory-Fejér interpolation on polydisks*, Math. Res. Lett. **7** (2000), no. 1, 25–34. MR 1748285 (2001j:47017)

[FF90] Ciprian Foias and Arthur E. Frazho, *The commutant lifting approach to interpolation problems*, Operator Theory: Advances and Applications, vol. 44, Birkhäuser Verlag, Basel, 1990. MR 1120546 (92k:47033)

[Hol01] John A. Holbrook, *Schur norms and the multivariate von Neumann inequality*, Recent advances in operator theory and related topics (Szeged, 1999), Oper. Theory Adv. Appl., vol. 127, Birkhäuser, Basel, 2001, pp. 375–386. MR 1902811 (2003e:47016)

[HWH14] Ming-Hsiu Hsu, Lih-Chung Wang, and Zhen He, *Interpolation problems for holomorphic functions*, Linear Algebra Appl. **452** (2014), 270–280. MR 3201101

[KP63] A. Korányi and L. Pukánszky, *Holomorphic functions with positive real part on polycylinders*, Trans. Amer. Math. Soc. **108** (1963), 449–456. MR 0159029 (28 #2247)

[Mis84] Gadadhar Misra, *Curvature inequalities and extremal properties of bundle shifts*, J. Operator Theory **11** (1984), no. 2, 305–317. MR 749164 (86h:47057)

[Mis94] G. Misra, *Completely contractive Hilbert modules and Parrott's example*, Acta Math. Hungar. **63** (1994), no. 3, 291–303. MR 1261472 (96a:46097)

[MNS90] Gadadhar Misra and N. S. Narasimha Sastry, *Completely bounded modules and associated extremal problems*, J. Funct. Anal. **91** (1990), no. 2, 213–220. MR 1058968 (91m:46074)

[MP93] Gadadhar Misra and Vishwambhar Pati, *Contractive and completely contractive modules, matricial tangent vectors and distance decreasing metrics*, J. Operator Theory **30** (1993), no. 2, 353–380. MR 1305512 (95j:46092)

[Nik86] N. K. Nikolʹskiĭ, *Treatise on the shift operator*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 273, Springer-Verlag, Berlin, 1986, Spectral function theory, With an appendix by S. V. Hruščev [S. V. Khrushchëv] and V. V. Peller, Translated from the Russian by Jaak Peetre. MR 827223 (87i:47042)

[Par70] Stephen Parrott, *Unitary dilations for commuting contractions*, Pacific J. Math. **34** (1970), 481–490. MR 0268710 (42 #3607)

[Par78] ______, *On a quotient norm and the Sz.-Nagy - Foiaş lifting theorem*, J. Funct. Anal. **30** (1978), no. 3, 311–328. MR 518338 (81h:47006)

[Pau92] Vern I. Paulsen, *Representations of function algebras, abstract operator spaces, and Banach space geometry*, J. Funct. Anal. **109** (1992), no. 1, 113–129. MR 1183607 (93h:46001)

[Pau02] Vern Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, vol. 78, Cambridge University Press, Cambridge, 2002. MR 1976867 (2004c:46118)

[Pis01] Gilles Pisier, *Similarity problems and completely bounded maps*, expanded ed., Lecture Notes in Mathematics, vol. 1618, Springer-Verlag, Berlin, 2001, Includes the solution to "The Halmos problem". MR 1818047 (2001m:47002)

[Pis03] ______, *Introduction to operator space theory*, London Mathematical Society Lecture Note Series, vol. 294, Cambridge University Press, Cambridge, 2003. MR 2006539 (2004k:46097)

[Rud08] Walter Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Classics in Mathematics, Springer-Verlag, Berlin, 2008, Reprint of the 1980 edition. MR 2446682 (2009g:32001)

[Var74] N. Th. Varopoulos, *On an inequality of von Neumann and an application of the metric theory of tensor products to operators theory*, J. Functional Analysis **16** (1974), 83–100. MR 0355642 (50 #8116)

[Var76] ______, *On a commuting family of contractions on a Hilbert space*, Rev. Roumaine Math. Pures Appl. **21** (1976), no. 9, 1283–1285. MR 0430824 (55 #3829)

[vN51] Johann von Neumann, *Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes*, Math. Nachr. **4** (1951), 258–281. MR 0043386 (13,254a)

[Woe02] Hugo J. Woerdeman, *Positive Carathéodory interpolation on the polydisc*, Integral Equations Operator Theory **42** (2002), no. 2, 229–242. MR 1870441 (2002i:47028)

[You88] Nicholas Young, *An introduction to Hilbert space*, Cambridge Mathematical Textbooks, Cambridge University Press, Cambridge, 1988. MR 949693 (90e:46001)